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Rectifiable sets and the Traveling Salesman Problem*

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§1. Introduction

Let $K \subset \mathbb{C}$ be a bounded set. In this paper we shall give a simple necessary and sufficient condition for K to lie in a rectifiable curve. We say that a set is a rectifiable curve if it is the image of a finite interval under a Lipschitz mapping. Recall that for a connected set $\Gamma \subset \mathbb{C}$, Γ is a rectifiable curve (not necessarily simple) if and only if $l(\Gamma) < \infty$, where $l(\cdot)$ denotes one dimensional Hausdorff measure. This classical result follows from the fact that on any finite graph there is a tour which covers the entire graph and which crosses each edge (but not necessarily each vertex!) at most twice. If K is a finite set then we are essentially reduced to the classical Traveling Salesman Problem (TSP): Compute the length of the shortest Hamiltonian cycle which hits all points of K. This is the same, up to a constant multiple, as asking for the infimum of $l(\Gamma)$ where Γ is a curve, $K \subset \Gamma$. (Such a Γ is called a spanning tree in TSP theory.) For infinite sets K, we cannot hope in general to have K be a subset of a Jordan curve. What we should therefore look at is connected sets which contain K.

Let $\Gamma_{\min}$ be the shortest (minimal) spanning tree. Then we cannot possibly solve our problem for sets K of infinite cardinality if we cannot find Γ, $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$, for any finite set K. (Here and throughout the paper C, C_0, C_1, c_0, etc. denote various universal constants.) While there are several algorithms for computing $l(\Gamma_{\min})$, these algorithms work for finite graphs satisfying the triangle inequality, and do not use the Euclidean properties of K. (See [13] for an excellent discussion of some of these algorithms.) Therefore these methods cannot solve our problem for general infinite K. We present a method which is a minor modification of a well-known algorithm ("Farthest Insertion" – see [13]) which yields a Γ with $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$. The Farthest Insertion algorithm has been extensively studied with large numerical calculations on computers, and is experimentally good in the sense that the Γ produced satisfy $l(\Gamma) \leqq C_0 l(\Gamma_{\min})$ for all examples which have

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been calculated. If K contains N points our method could be modified to calculate (up to a constant multiple) $l(\Gamma_{\text{opt}})$ in time $0(N \log N)$, which is what one would expect and which is available for other algorithms [17]. We can thus say that, in a certain sense, our theorem gives a geometric "solution" to the TSP.

A square $Q \subset \mathbb{C}$ is a dyadic square if $Q = [j2^{-n}, (j+1)2^{-n}] \times [k2^{-n}, (k+1)2^{-n}]$, where $j, k, n \in \mathbb{Z}$. We denote by $l(Q) = 2^{-n}$ the sidelength of Q. For $\lambda > 0$ we denote by λQ the square with the same center as Q, with sidelength $\lambda l(Q)$, and with sides parallel to the axes. For Q a dyadic square let S_Q be an infinite strip (or line in the degenerate case) of smallest possible width which contains $K \cap 3Q$, and let $\omega(Q)$ denote the width of S_Q. We then define

$$\beta_K(Q) = \frac{\omega(Q)}{l(Q)},$$

so that $\beta_K(Q)$ measures in a scale invariant way ($0 \leqq \beta_K(Q) \leqq 3$) the deviation of K from a straight line near Q on scale $l(Q)$. Our main result states that K is contained in a rectifiable curve Γ if and only if

$$\beta^2(K) \equiv \sum_Q \beta_K^2(Q)\, l(Q) < \infty \tag{1.1}$$

where the sum is taken over all dyadic squares (or equivalently over all dyadic squares Q with $l(Q) \leqq$ diameter (K)). Furthermore, the shortest possible Γ has length comparable to diameter (K) plus the sum in (1.1). This is an elementary (if quite lengthy) computation with the Pythagorean theorem if $\beta_K(Q) < \varepsilon_0$ for all Q (ε_0 sufficiently small) or if K already lies in a Lipschitz curve (defined in Section 2). However, if $\beta_K(Q)$ is large for many Q (K is "dispersed") there are technical difficulties to be overcome. Conversely, if Γ is a Lipschitz curve it is not hard to prove (and is already in [9]) that $\beta^2(\Gamma)$ is bounded by a multiple of $l(\Gamma)$.

Theorem 1. *If $\Gamma \subset \mathbb{C}$ is connected, then*

$$\beta^2(\Gamma) \leqq C_0\, l(\Gamma)\,. \tag{1.2}$$

Conversely, if $\beta^2(K) < \infty$ there is a connected set Γ, $K \subset \Gamma$, such that

$$l(\Gamma) \leqq (1+\delta)\,\text{diameter}\,(K) + C(\delta)\,\beta^2(K)\,. \tag{1.3}$$

Corollary 1. *If K is an analytic set and $l(K) < \infty$, then K is totally unrectifiable in the sense of Besicovitch* ([4], [5]) *if and only if*

$$\beta^2(E) = \infty$$

for every $E \subset K$ with $l(E) > 0$.

For certain purposes (see e.g. [1]) it is useful to know that the curve constructed for (1.3) can have extra properties. We will show that by taking δ large enough there is Γ, $K \subset \Gamma$, such that (1.3) holds, and

(1.4) If $z_1, z_2 \in \Gamma$ there is a connected set $\gamma \subset \Gamma$ such that $z_1, z_2 \in \gamma$ and $l(\gamma) \leqq C_0 |z_1 - z_2|$.

(1.5) If $K \cap Q \neq \phi$ and $l(Q) \leqq$ diameter (K), there is an infinite strip S_Q with width $\beta(Q)\, l(Q)$, with axis L_Q, $S_Q \cap K \cap Q \neq \phi$, and such that if P denotes orthogonal projection onto L_Q,

$$P(3Q) \subset P(\Gamma \cap S_Q)\,.$$

Condition (1.4) says that Γ is uniformly locally connected and (1.5) asserts that Γ "crosses" $S_Q \cap 3Q$.

To prove the first theorem we will use a result which is of some independent interest. An M Lipschitz domain of size one centered at the origin is a simply connected domain whose boundary is a Jordan curve described by $r(\theta)e^{i\theta}$, $0 \leqq \theta \leqq 2\pi$, where $\dfrac{1}{1+M} \leqq r(\theta) \leqq 1$, and where $|r(\theta_1) - r(\theta_2)| \leqq M\,|e^{i\theta_1} - e^{i\theta_2}|$. An M Lipschitz domain is a dilate and translate of one of the above domains. Lipschitz domains play a central rôle in modern one complex variable theory, and their behavior vis-à-vis Cauchy integrals and harmonic measure is well understood.

Theorem 2. *If Ω is a simply connected domain and $l(\partial\Omega) < \infty$, there is a rectifiable curve Γ such that $\Omega \backslash \Gamma = \bigcup_j \Omega_j$ is a decomposition of Ω into disjoint C_0 Lipschitz domains, and*

$$\sum_j l(\partial\Omega_j) \leqq C_0\, l(\partial\Omega)\,.$$

We thank Charles Pugh for pointing out that the above theorem fails if we replace C_0 by $\varepsilon_0 \ll 1$. Indeed if one tiles (in the above sense) the unit square by ε_0 Lipschitz domains Ω_j, $\sum l(\partial\Omega_j) = \infty$. Now let Γ be a bounded connected set and attach to Γ a line segment L and circle S. Applying Theorem 2 to $\mathbb{C}\backslash(\Gamma \cup L \cup S)$ we obtain

Corollary 2. *If Γ is connected there is a connected set $\tilde{\Gamma}$ such that $\Gamma \subset \tilde{\Gamma}$, $l(\tilde{\Gamma}) \leqq C_0\, l(\Gamma)$, every bounded component of $\mathbb{C}\backslash\tilde{\Gamma}$ is a C_0 Lipschitz domain, and the unbounded component of $\mathbb{C}\backslash\tilde{\Gamma}$ is the complement of a disk. Furthermore, if $x, y \in \tilde{\Gamma}$ there is a subarc $\gamma \subset \tilde{\Gamma}$ such that $x, y \in \gamma$ and $l(\gamma) \leqq C_1\,|x - y|$. The constant C_1 may be taken to be 3.*

The theorem's proof can be modified to yield further structure on Γ if (1.1) holds in some uniform sense. A curve Γ is said to be Ahlfors-David regular (AD) if

$$\sup_{\substack{z_0 \in \mathbb{C} \\ r > 0}} r^{-1}\, l(\{z \in \Gamma : |z - z_0| \leqq r\}) < \infty\,.$$

Then one can show that K lies in an AD curve if and only if

$$\sup_Q l(Q)^{-1} \sum_{Q' \subset Q} \beta_K^2(Q')\, l(Q') < \infty\,, \tag{1.6}$$

and this therefore gives us a characterization of AD curves. If Γ is an ε quasicircle passing through ∞, $\beta_\Gamma(Q) \leqq C_0 \varepsilon^{1/2}$ for all Q. Conversely, our construction can be

modified to show that if $\beta_K(Q) \leqq \varepsilon < \varepsilon_0$ for all Q, K is a subset of a $\delta(\varepsilon)$ quasicircle passing through ∞.

The history behind our proof of Theorem 1 is at first glance a bit surprising, in that it was discovered by a careful study of Cauchy integral operators on Lipschitz curves. If Γ is a Lipschitz curve, a famous theorem due to A.P. Calderón [2] (for small Lipschitz constants) and Coifman, McIntosh and Meyer [3] later, (for any Lipschitz curve) asserts that the Cauchy integral operator is bounded on the Lebesgue space $L^2(\Gamma)$. By now there are myriad proofs of this result. The theorem is very old when $\Gamma = \mathbb{R}$, and is equivalent there to the L^2 boundedness of the Hilbert transform. This is essentially easy because one can apply Plancherel's theorem. The author had attempted to find a proof of the result for Lipschitz curves by using the intuitive idea that a Lipschitz curve should look like a straight line at most places on most scales. This was accomplished in [9] by noting that (1.6) holds for Lipschitz curves.

Further reflection shows that the above connection between Cauchy integrals and the TSP should not be so surprising. Let K have finite one-dimensional Hausdorff measure, $l(K) < \infty$. An old problem is to decide whether $\gamma(K)$, the analytic capacity of K, is zero. In other words, is there a non-constant, bounded holomorphic function on $\mathbb{C}\backslash K$? (See [6] or [15] for more on $\gamma(K)$.) An easy necessary condition is that $l(K) > 0$. An old conjecture is that $\gamma(K) > 0$ if and only if $\mathrm{Fav}(K) > 0$, where $\mathrm{Fav}(\cdot)$ is Favard length,

$$\mathrm{Fav}(K) = \int_0^\pi l(K_\theta)\, d\theta\,,$$

and where K_θ is the orthogonal projection of K onto the line $\mathbb{R}e^{i\theta}$. This is known to be false for sets where K has non sigma finite length [11, 15], but is still open when $l(K) < \infty$. The idea Murai has used [11, 15] to attack this problem is to use deep estimates on the L^2 bounds for Cauchy integrals on Lipschitz curves. Now sets where $0 < l(K) < \infty$ and $\mathrm{Fav}(K) = 0$ play a central rôle in Besicovitch's theory of rectifiable sets (see e.g. [4, 5]). Besicovitch proved this occurs if and only if $l(K \cap \Gamma) = 0$ for every rectifiable curve Γ. Such sets are called totally irregular. Combining this with Calderón's theorem on the Cauchy integral [2], it follows that $l(K) < \infty$ and $\mathrm{Fav}(K) > 0$ imply $\gamma(K) > 0$. This connection between Cauchy integrals, geometric measure theory, and the TSP seems all the more natural when one considers that the Cauchy integral also plays a vital part in Borsuk's proof of the Jordan curve theorem. In [1] Christopher Bishop and the author use Theorem 1 and the machinery developed for Cauchy integrals to settle some conjectures about harmonic measure.

The paper is organized as follows. In Section 2 we use an argument with conformal mappings to prove Theorem 2. Section 3 uses Theorem 2 to give a proof of (1.2). Section 4 is devoted to a construction of the curve Γ which satisfies (1.3)–(1.5). Section 5 is an appendix where we give an elementary proof that (1.2) holds for Lipschitz curves. That result (first proved by Fourier analysis in [9]) is used in Section 2.

The author is grateful to Stephen Semmes for many conversations on condition (1.1). I believed at one time that I had a counter-example to (1.3). When Stephen

Semmes repeatedly asked me to write it down, I found an error, and Theorem 1 soon appeared.

§2. Proof of Theorem 2

We assume the reader is familiar with corona type constructions as presented e.g. in the book of Garnett [7]. Let $F: \mathbb{D} \to \Omega$ be any choice of a Riemann map of the unit disk onto Ω. Since the theorem is trivial if $l(\partial\Omega) = \infty$, we assume $l(\partial\Omega) < \infty$ so that $F'(z)$ is in the Hardy space H^1. (See [11].) Let $G(z) = (F'(z))^{1/2}$ so that $G \in H^2$, and apply either Green's formula or Plancherel to obtain

$$\int_{\mathbb{T}} |G(e^{i\theta}) - G(0)|^2 \, d\theta = \iint_{\mathbb{D}} |G'(z)|^2 \log \frac{1}{|z|} \, dxdy \, .$$

From this we see that if we set $F' = G^2 = e^{\varphi}$,

$$\iint_{\mathbb{D}} |F'(z)|\,|\varphi'(z)|^2 \log \frac{1}{|z|} \, dxdy \leqq 4 \int_{\mathbb{T}} |F'(e^{i\theta})| \, d\theta \tag{2.1}$$

$$\leqq 8\, l(\partial\Omega) \, .$$

Now it is well known that φ is in the Bloch space B and of norm at most 6, i.e.

$$|\varphi'(z)| \leqq 6(1 - |z|^2)^{-1}, \qquad z \in \mathbb{D} \, . \tag{2.2}$$

See e.g. [16]. Let us decompose T into dyadic arcs, i.e. arcs of length $\pi 2^{-n}$. For such a dyadic arc I we associate to it a dyadic "square"

$$Q = Q_I = \left\{ z : \frac{z}{|z|} \in I, \, 1 - \pi 2^{-n} \leqq |z| \leqq 1 \right\} .$$

Let $T_Q = \{z \in Q : 1 - 2\pi^{-n} \leqq |z| \leqq 1 - \pi 2^{-n-1}\}$ denote the "top half" of Q. Let z_Q be the center of T_Q.

Fix a dyadic square Q and perform the following stopping time argument. If there is $z \in T_Q$ with $|\varphi(z) - \varphi(z_Q)| \geqq \varepsilon$, stop and let $\mathscr{D}_Q = T_Q$. In this case we say Q is of type 0. Otherwise let $Q_1, Q_2, \ldots$ be those dyadic squares inside Q which satisfy

$$\sup_{z \in T_{Q_j}} |\varphi(z) - \varphi(z_Q)| \geqq \varepsilon \, ,$$

and define

$$\mathscr{D}_Q = Q \setminus \bigcup_{j=1}^{\infty} Q_j \, .$$

Then if $\mathscr{D}_Q$ is not of Type 0,

$$|\varphi(z) - \varphi(z_Q)| \leqq \varepsilon, \qquad z \in \mathscr{D}_Q \, . \tag{2.3}$$

By the construction of $\mathscr{D}_Q$ we see that $\partial\mathscr{D}_Q \equiv \gamma_Q$ is a chord arc domain:

If $z_1, z_2 \in \gamma_Q$, there is an arc $\gamma \subset \gamma_Q$ joining z_1 to z_2 and of length $l(\gamma) \leqq 6\,|z_1 - z_2|$.

Using this procedure we tile $\mathbb{D}$ by regions $\mathscr{D}_Q$ via the usual stopping time arguments. If a region $\mathscr{D}_Q$ so formed is not of Type 0, we say $\mathscr{D}_Q$ is of Type 1 if $l(T \cap \gamma_Q) \geqq \frac{1}{2} l(T \cap \partial Q)$, and we say $\mathscr{D}_Q$ is of Type 2 otherwise. Then if $\mathscr{D}_Q$ is of Type 1 or 2,

$$|F'(z) - F'(z_Q)| \leqq 2\varepsilon |F'(z_Q)|$$

whenever $z \in \mathscr{D}_Q$.

We first bound the Type 0 regions. By normal families and (2.2),

$$\int_{\gamma_Q} |F'(z)|\, ds(z) \leqq C \iint_{T_Q} |F'(z)|\, |\varphi'(z)|^2 \log \frac{1}{|z|}\, dxdy\,.$$

whenever $\mathscr{D}_Q = T_Q$ is of Type 0. Consequently, inequality (2.1) yields

$$\sum_{\text{Type 0}} \int_{\gamma_Q} |F'(z)| ds(z) \leqq C l(\partial\Omega)\,,$$

because the regions $\mathscr{D}_Q$ are disjoint. Now let N_0 be a suitably large integer (in fact, $N_0 = 2$ will do) and divide each $\mathscr{D}_Q = T_Q \in$ Type 0 into 4^{N_0} "squares" of essentially equal sidelength. Then if $\tilde{\mathscr{D}}_Q$ is any of these "squares", (2.2) shows that $F(\tilde{\mathscr{D}}_Q)$ is a C_0 Lipschitz domain. By our last estimate,

$$\sum_{\mathscr{D}_Q \in \text{Type 0}} \sum_{\tilde{\mathscr{D}}_Q} l(\partial F(\tilde{\mathscr{D}}_Q)) \leqq C_0\, l(\partial\Omega)\,.$$

We now turn to the regions of Type 1. Since the sets $T \cap \partial Q_j$ and $T \cap \partial Q_k$ can intersect in at most two points if $j \neq k$, and since (2.3) holds,

$$\begin{aligned}
&\sum_{\text{Type 1}} \int_{\gamma_Q} |F'(z)| ds(z) \\
&\leqq \sum_{\text{Type 1}} 12(1 + 2\varepsilon) \int_{T \cap \gamma_Q} |F'(z)|\, ds(z) \\
&\leqq 12(1 + 2\varepsilon) \int_T |F'(z)|\, ds(z) \\
&= 12(1 + 2\varepsilon)\, l(\partial\Omega)\,. \qquad (2.4)
\end{aligned}$$

The Type 2 regions are a little trickier to bound. The following is the main idea of this section: Use the L^2 bounds of (2.1) to bound the Type 2 regions. Let $I_1, I_2, \ldots$ be the horizontal line segments in $\gamma_Q \setminus \partial Q$. Then by hypothesis,

$$\sum_{j=1}^{\infty} l(I_j) \geqq \frac{1}{12}\, l(\gamma_Q)\,. \qquad (2.5)$$

By normal families and estimate (2.2) there is $\delta > 0$ such that

$$l(\{z \in I_j : |\varphi(z) - \varphi(z_Q)| \geqq \delta\}) \geqq \delta l(I_j)\,. \qquad (2.6)$$

(Because $\sup_{z \in T(Q_j)} |\varphi(z) - \varphi(z_Q)| \geqq \varepsilon$.) Now let ω denote harmonic measure for the

region $\mathscr{D}_Q$ with respect to the base point z_Q, and let $g(z)$ denote Green's function for $\mathscr{D}_Q$ with pole at z_Q so that

$$d\omega = \frac{1}{2\pi}\frac{\partial g}{\partial n}\,ds\,.$$

Then since $\mathscr{D}_Q$ is a chord-arc domain, the results of Jerison and Kenig [8] show that ω is an A_∞ weight, i.e. there are constants $A, \eta, \eta' > 0$ so that

$$A^{-1}\left(\frac{l(E)}{l(\gamma_Q)}\right)^{\eta} \leqq \omega(E) \leqq A\left(\frac{l(E)}{l(\gamma_Q)}\right)^{\eta'} \tag{2.7}$$

for any Borel set $E \subset \gamma_Q$. (The usual definition of A_∞ uses only the second inequality, but it is well known that this is equivalent to the first [7].) Combining (2.3) with (2.5)–(2.7) we see that

$$\int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega \geqq c\,|F'(z_Q)|\,,$$

where $c = c(\delta, A, \eta) > 0$. Green's formula is valid for chord arc domains, so that

$$2\pi \int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega = \iint_{\mathscr{D}_Q} |G'(z)|^2\,g(z)\,dxdy\,.$$

An elementary argument with the maximum principle shows that if $I = T \cap \partial Q$,

$$g(z) \leqq C\,l(I)^{-1} \log\frac{1}{|z|}$$

whenever $z \in \mathscr{D}_Q$ and $|z - z_Q| \geqq \frac{1}{16}\,l(I)$. Applying the maximum principle again we obtain

$$\begin{aligned}\int_{\gamma_Q} |F'(z)|\,ds(z) &\leqq C\,l(\gamma_Q) \int_{\gamma_Q} |G(z) - G(z_Q)|^2\,d\omega \\ &\leqq C'\,l(\gamma_Q) \iint_{\mathscr{D}_Q} |G'(z)|^2\,l(\gamma_Q)^{-1} \log\frac{1}{|z|}dx\,dy \\ &= C'' \iint_{\mathscr{D}_Q} |F'(z)|\,|\varphi'(z)|^2 \log\frac{1}{|z|}\,dx\,dy\,.\end{aligned}$$

Since the regions $\mathscr{D}_Q$ are disjoint, it follows from (2.1) that

$$\begin{aligned}\sum_{\text{Type 2}} \int_{\gamma_Q} |F'(z)|\,ds(z) &\leqq C \iint_{\mathscr{D}} |F'(z)|\,|\varphi'(z)|^2 \log\frac{1}{|z|}dx\,dy \\ &\leqq C'\,l(\partial\Omega)\,.\end{aligned} \tag{2.8}$$

This almost finishes the proof of the theorem, because if ε is small enough each domain $F(\mathscr{D}_Q)$ will have boundary a (7) chord-arc curve. (This follows immediately from (2.3).) Furthermore by estimates (2.4) and (2.8) on the Type 1 and Type 2 domains,

$$\sum l(\partial F(\mathscr{D}_Q)) \leqq C\,\ell\,(\partial\Omega)\,.$$

We thus need only show that each Type 1 or 2 region $\mathscr{D}_Q$ can be decomposed into disjoint C_0 Lipschitz domains Ω_j with $\sum_j l(\partial\Omega_j) \leqq C_0 l(\gamma_Q)$. For then $F(\Omega_j)$ will be a C_1 Lipschitz domain if ε is small enough, and

$$\sum_j l(\partial F(\Omega_j)) \leqq C\, l(\partial F(\mathscr{D}_Q)) .$$

The construction is a little easier to describe on the upper half plane $\mathbb{R}^2_+ = \{z = x + iy: y > 0\}$. We are given a dyadic square $Q \subset \mathbb{R}^2_+$ with side $I \subset \mathbb{R}$. The region $\mathscr{D}_Q$ is constructed by removing from Q disjoint squares $Q_1, Q_2, \ldots$ with (disjoint) sides $I_j \subset \mathbb{R}$. Let $\tilde{I}_j$ denote the top side of Q_j. We now build a Cantor type tree T_j. Put $\tilde{I}_j$ in T_j and add on line segments on the left and right of I_j down to $\mathbb{R}$ and of length $\sqrt{2}\, l(I_j)$. We obtain a nice "tree" T_j^1 with two "roots" of length $\sqrt{2}\, l(I_j)$ touching $\mathbb{R}$ at angle $\pi/4$. Let z_1 and z_2 be the two points in T_j^1 of height $1/4\, l(I_j)$ so that $\operatorname{Re} z_j = x_j = k_j(1/4\, l(I_j))$, $k_j \in \mathbb{Z}$, $j = 1, 2$. (In other words, x_1 and x_2 are appropriate dyadic rationals.) Attach to z_1 and z_2 the line segments to $\mathbb{R}$ (of angle $\pi/4$ with $\mathbb{R}$) which are not already in T_j^1 and call this T_j^2. T_j^2 is obtained by sprouting the roots of T_j^1. Continue sprouting, by dropping by a factor of $1/4$ each time and obtain $T_j^1, T_j^2, \ldots, T_j^n, \ldots$. Then T_j^n has 2^n "roots" of length $\sqrt{2}\, 4^{-n}\, l(I_j)$ and each terminates at a point $k\, 4^{-n}\, l(I_j)$ where $k \in \mathbb{Z}$. Setting $T_j = \bigcup_{k=1}^{\infty} T_j^k$ we have

$$l(T_j) = (1 + 3\sqrt{2})\, l(I_j) .$$

It is an exercise to see that every component of $\mathscr{D}_Q \setminus \bigcup_j T_j$ is a C_0 Lipschitz domain. □

We remark that the idea of using Littlewood–Paley estimates to impose bi-Lipschitz structures on sets has been used before in [10] to give a certain quantitative version of Sard's theorem.

§3. Proof of (1.2)

We may assume that $l(\Gamma) < \infty$ and the conclusions of Corollary 2 hold for Γ, because if $\Gamma \subset \tilde{\Gamma}$, $\beta^2(\Gamma) \leqq \beta^2(\tilde{\Gamma})$. We denote by Ω_j the components of $\mathbb{C} \setminus \Gamma$ and set $\Gamma_j = \partial\Omega_j$, $d_j =$ diameter (Γ_j). For a dyadic square Q let $\mathscr{F}(Q) = \{\Gamma_j: \Gamma_j \cap 4Q \neq \phi, d_j \geqq l(Q)\}$ and let $G(Q) = \{\Gamma_j: \Gamma_j \cap 5Q \neq \phi, d_j < l(Q)\}$. Also let Q^* be the dyadic double of Q.

Lemma 3.1.

$$\beta_\Gamma^2(Q) \leqq C_1 \sum_{\mathscr{F}(Q)} \beta_{\Gamma_j}^2(Q^*) + C_1\, l(Q)^{-2} \sum_{G(Q)} \text{Area}\,(\Omega_k) .$$

Proof. The lemma is immediate if $\mathscr{F}(Q) = \phi$, for then either $\beta(Q) = 0$, or $\sum_{G(Q)} \text{Area}\,(\Omega_k) \geqq 9l(Q)^2$. So suppose $\Gamma_0 \in \mathscr{F}(Q)$. By scaling we may assume $l(Q) = 1$.

Let L be a line such that

$$d \equiv \sup_{z \in \Gamma_0 \cap 5Q} \text{distance}\,(z, L) \leqq \beta_{\Gamma_0}(Q^*)\, l(Q^*)\,.$$

Let $z_0 \in \Gamma \cap 3Q$ maximize distance (z, Γ_0) and let distance $(z_0, \Gamma_0) = d_0$. Denote by z_1 the closest point in Γ_0 to z_0, and let $I = [z_0, z_1]$. We define $z_2 = 1/2(z_0 + z_1)$ to be the midpoint of I.

Case 1. $\mathcal{F}(Q)$ contains three or more curves. Then there is $\Gamma_j \in \mathcal{F}(Q)$ such that $\beta_\Gamma^2(Q^*) \geqq C_1$, so

$$\sum_{\mathcal{F}(Q)} \beta_{\Gamma_j}^2(Q^*) \geqq C_1\,.$$

Case 2. $\mathcal{F}(Q)$ contains only Γ_0. Then $B\left(z_2, \frac{d_0}{2}\right) \subset \bigcup_{G(Q)} \bar{\Omega}_k$. Consequently,

$$\begin{aligned}\beta_\Gamma^2(Q) &\leqq (d + d_0)^2 \\ &\leqq 2\beta_{\Gamma_0}^2(Q^*) + 2d_0^2 \\ &\leqq 2\beta_{\Gamma_0}^2(Q^*) + \frac{8}{\pi} \sum_{G(Q)} \text{Area}\,(\Omega_k)\,.\end{aligned}$$

Case 3. $\mathcal{F}(Q) = \{\Gamma_0 \Lambda \Gamma_j\}$.

We may as well assume $\beta_{\Gamma_j}(Q^*) < \varepsilon_0, j = 0, 1$, for otherwise there is nothing to prove. Let $d_1 = \sup_{z \in \Gamma_1 \cap 4Q} \text{distance}\,(z, \Gamma_0)$. Then if ε_0 is small enough $\Gamma \cap 3Q$ is trapped between Γ_0 and Γ_1, and

$$\beta_\Gamma(Q) \leqq \beta_{\Gamma_0}(Q^*) + \beta_{\Gamma_1}(Q^*)) + d_1\,.$$

Since Γ_1 is a C_0 Lipschitz curve there is $z_3 \in 9/2Q \setminus (\Omega_0 \cup \Omega_1)$ such that distance $(z_3, \Gamma_j) \geqq c_0 d_1$, $j = 0, 1$. But then

$$\begin{aligned}\beta_\Gamma^2(Q) &\leqq C_0(\beta_{\Gamma_0}^2(Q^*) + \beta_{\Gamma_1}^2(Q^*)) + C_0 d_1^2 \\ &\leqq C_0(\beta_{\Gamma_0}^2(Q^*) + \beta_{\Gamma_0}^2(Q^*)) + C_1 \sum_{G(Q)} \text{Area}\,(\Omega_k)\,.\end{aligned}$$

It is now an easy matter to finish the proof of (1.2). By the results of [9] (or our section 5),

$$\sum_Q \beta_{\Gamma_j}^2(Q)\, l(Q) \leqq C_1\, l(\Gamma_j)$$

for any C_0 Lipschitz curve. By Lemma 3.1 it is enough to estimate the sum

$$\sum_Q l(Q)^{-1} \sum_{G(Q)} \text{Area}\,(\Omega_k)\,.$$

Now for each $n \in \mathbb{Z}$ such that $d_j < 2^{-n}$ there are at most C_1 squares Q such that $l(Q) = 2^{-n}$ and $\Gamma_j \in G(Q)$. Consequently, we may estimate the above sum by

reversing the order of summation to obtain

$$\sum_{\Gamma_k} \text{Area}\,(\Omega_k) \sum_{\substack{Q \\ \Omega_k \in G(Q)}} l(Q)^{-1}$$

$$\leqq C \sum_{\Gamma_k} \text{Area}\,(\Omega_k) \sum_{m=0}^{\infty} (2^m d_k)^{-1}$$

$$\leqq 2C \sum_{\Gamma_k} \text{Area}\,(\Omega_k)\, d_k^{-1}$$

$$\leqq C' \sum l(\Gamma_k) \leqq 2C'\, l(\Gamma)\,.$$

§4. Construction of Γ. (Proof of (1.3)–(1.5).)

We build sets $\mathscr{L}_0, \mathscr{L}_1, \ldots, \mathscr{L}_n \subset K$ with the following properties:

$$|z_j - z_k| \geqq 2^{-n}, \qquad z_j, z_k \in \mathscr{L}_n, \qquad j \neq k\,.$$

and

$$\inf_{z_j \in \mathscr{L}_n} |z - z_j| \leqq 2^{-n}, \qquad z \in K\,.$$

These sets may need to be slightly perturbed at various stages of the construction, but the two properties listed above will still hold. We may assume $K \subset [0, 1]^2$, and by scaling we may also assume $\mathscr{L}_0 = \{z_0, z_1\}$ where $|z_0 - z_1| = \sup_{z, w \in K} |z - w|$. We define $\Gamma_0 = [2z_0 - z_1, 2z_1 - z_0]$ to be a line segment containing z_0, z_1, and extending beyond those points. We let $A \geqq 1$ be a constant to be fixed later, $A = 2^{k_0}$, and then let $\varepsilon_0 > 0$ be small. The value of ε_0 is determined later. Suppose by induction that $\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}$ have been formed and let $x_0 \in \mathscr{L}_n \backslash \mathscr{L}_{n-1}$. If $x_0 \in \Gamma_{n-1}$ we do no construction about x_0. Let Q be that dyadic cube containing x_0 with $l(Q) = A2^{-n}$. We call the collection of all such cubes $\mathscr{D}_n$. By a translation and rotation we may assume $x_0 \geqq 0$ and its nearest neighbor in $\mathscr{L}_{n-1}$ is the origin. Let $W = \{z\colon 0 < |z| \leqq A2^{-n+1}, |\arg z| \leqq 2\pi/3\}$, and let $W^* = \{z\colon 0 < |z| \leqq A2^{-n+1}, |\pi\text{-arg}\, z| \leqq 2\pi/3\}$, so that $W \cup W^* = \mathscr{D} = \{z\colon 0 < |z| \leqq A2^{-n+1}\}$. We assume by induction that the following properties hold:

(P1) Let $\{y_1, \ldots, y_N\} = \mathscr{L}_{n-1} \cap \{\mathscr{D} \cup \{0\}\}$ and arrange the points so that $\text{Re}\, y_1 \leqq \text{Re}\, y_2 \leqq \ldots \leqq \text{Re}\, y_N$. Then Γ_{n-1} contains the segments $[y_j, y_{j+1}]$, $1 \leqq N - 1$.

(P2) If $x_0 \notin \Gamma_{n-1}$ and $\mathscr{L}_{n-1} \cap W = \phi$ there is θ, $|\theta| \leqq 2\pi/3$, such that $[0, A2^{-n+1} e^{i\theta}] \subset \Gamma_{n-1}$. If $x_0 \notin \Gamma_{n-1}$ and $\mathscr{L}_{n-1} \cap W^* = \phi$, there is ψ, $|\psi - \pi| \leqq 2\pi/3$, such that $[0, A2^{-n+1} e^{i\psi}] \subset \Gamma_{n-1}$.

Case 1. $\beta(Q) \geqq \varepsilon_0$. Connect x_0 to all points in $\mathscr{L}_n \cap \{|x - x_0| \leqq C2^{-n}\}$ by (straight) line segments. Also add on the line segment $[0, 4Ax_0]$. Then the amount

of length added to Γ_{n-1} is bounded by

$$C \sum_{Q \in \mathcal{Q}_n} \ell(Q) \leqq C' \sum_{Q \in \mathcal{Q}_n} \beta^2(Q)\, l(Q)$$

because each Case 1 Q has $\beta(Q)^2 \geqq \varepsilon_0^2$.

For the rest of the cases we assume $\beta(Q) < \varepsilon_0$.

Case 2. $\mathcal{L}_{n-1} \cap W \neq \phi$, $\mathcal{L}_{n-1} \cap W^* \neq \phi$. Let $y_1 \in W$ minimize $|z|$, $z \in \mathcal{L}_{n-1} \cap W$, and let $y_{-1} \in W^*$ minimize $|z|$, $z \in \mathcal{L}_{n-1} \cap W^*$. Let $\{x_1, \ldots, x_M\}$ be those points in $\mathcal{L}_n \cap W$ such that $\operatorname{Re} y_{-1} \leqq \operatorname{Re} x_j \leqq \operatorname{Re} y_1$, and label the points so that $\operatorname{Re} x_1 \leqq \operatorname{Re} x_2 \leqq \ldots \leqq \operatorname{Re} x_M$. Replace the segments $[y_{-1}, 0]$, $[0, y_1] \subset \Gamma_{n-1}$ with $[x_1, x_2], \ldots, [x_{M-1}, x_M]$. Then by the Pythagorean theorem, the amount of length added to Γ_n is bounded by

$$C\beta^2(Q)\, l(Q)\,.$$

Since $\beta(Q) < \varepsilon_0$, properties (P1) and (P2) are maintained.

Case 3. $\mathcal{L}_{n-1} \cap W \neq \phi \cong \mathcal{L}_{n-1} \cap W^* = \phi$, $\mathcal{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} = \phi$. Let $y_1 \in \mathcal{L}_{n-1} \cap W$ minimize $|z|$, $z \in \mathcal{L}_{n-1} \cap W$. Then $[0, y_1] \subset \Gamma_{n-1}$. Let $\{0, x_1, x_2, \ldots, x_M, y_1\}$ be the points in $\mathcal{L}_n \cap W$ between 0 and y_1, arranged by increasing real parts. Replace $[0, y_1]$ by $[0, x_1], [x_1, x_2], \ldots, [x_M, y_1]$. Then by Pythagoras the amount of length added is bounded by

$$C\beta^2(Q)\, l(Q)\,,$$

and properties (P1) and (P2) are maintained at $x_1, \ldots, x_M$ because $\beta(Q) < \varepsilon_0$. We now add on some line segments near 0 and y_1 to assure properties (P1) and (P2) hold there. First suppose $x_{-1} \in K \cap W^*$ maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq A2^{-n}\}$. If $|x_{-1}| < 8A^{-1}2^{-n}$ add to Γ_n the line segment $[-2^{-n+1}, 0]$. If $|x_{-1}| \geqq 8A^{-1}2^{-n}$, add to Γ_n the line segment $\left[2^{-n+1}\frac{x_{-1}}{|x_{-1}|}, 0\right]$. In either case the amount of length added is bounded by 2^{-n+1}. Let $z_0, z_1, \ldots, z_N = x_1$ be the points in $\mathcal{L}_{n+k_0} \cap \{|z| \leqq 2^{-n}\}$, arranged by increasing real parts, where $2^{k_0} = 2A^2$. We add on to Γ_n the line segments $[z_0, z_1], \ldots, [z_{N-1}, z_N]$, so that at stages $n+1$, $n+2, \ldots, n+k_0-1$ no constructions need be performed in $\{|z| \leqq 2^{-n}\}$. Since $\beta(Q) < \varepsilon_0$, properties (P1), (P2) are preserved for $z_1, \ldots, z_N$ at stage $n+k_0$ by the choice of $[-2^{-n+1}, 0]$ [respectively $2^{-n+1}\frac{x_{-1}}{|x_{-1}|}, 0$]. A similar construction is performed at y_1.

Let $E = [0, A2^{-n+1}e^{i\theta}] \subset \Gamma_{n-1} \cap W^*$ be the line segment assured by hypothesis (P2) and let

$$I_Q = [A2^{-n-1}e^{i\theta}, A2^{-n}e^{i\theta}]\,.$$

Then the construction will show that I_Q is not altered at any future stage, $I_Q \subset \bigcap_{k=0}^{\infty} \Gamma_{n+k}$. If A is large enough,

(4.1) The amount of length added is bounded by $1/2\, l(I_Q)$.

Furthermore, if I_Q and $I_{Q'}$ are any intervals so formed at any stage of the construction,

$$I_Q \cap I_{Q'} = \phi \,. \tag{4.2}$$

Case 4. $\mathscr{L}_{n-1} \cap W \neq \phi$, $\mathscr{L}_{n-1} \cap W^* = \phi$, $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} \neq \phi$. First suppose $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\} = \{x_{-1}\}$. We may assume, by changing $\mathscr{L}_n$ if necessary, that x_{-1} maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$. The construction is as in Case 3, but we add the segment $[2x_{-1}, 0]$. Then estimates (4.1) and (4.2) hold and as in the argument of Case 3 properties (P1) and (P2) will hold at future stages. If $\mathscr{L}_n \cap W^* \cap \{|x| \leqq 2^{-n+1}\}$ contains two points x_{-1}, x_{-2}, we may assume x_{-1} maximizes $|z|$, $z \in K \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$. Then we add $[2x_{-1}, x_{-1}]$, and $[x_{-1}, x_{-2}]$, $[x_{-2}, 0]$. Estimates (4.1) and (4.2) hold as do properties (P1) and (P2) at future stages. The case where $\mathscr{L}_n \cap W^* \cap \{|z| \leqq 2^{-n+1}\}$ contains two points is treated similarly.

Case 5. $\mathscr{L}_{n-1} \cap W = \phi \cap \mathscr{L}_{n-1} \cap W^* \neq \phi$. We assume x_0 maximizes $|z|$, $z \in K \cap W \cap \{|z| \leqq 2^{-n+1}\}$. Let $\{y_1, y_2, \ldots, y_N = x_0\}$ be the points in $(\mathscr{L}_n \cap W^*) \cup \{|z| \leqq 2^{-n+1}\}$ arranged by increasing real parts, and as in Case 3 replace arcs of Γ_{n-1} in that region by $[y_1, y_2], \ldots, [y_{N-1}, y_N]$. As in Case 3 we add on $[x_0, 2x_0]$ and line segments in $W \cap \{|z| \leqq 2^{-n+1}\}$ so that $\mathscr{L}_{n+k_0} \cap W \cap \{|z| \leqq 2^{-n+1}\} \subset \Gamma_n$. We also choose I_Q as in Case 3. Then (4.1) and (4.2) hold and as in Case 3, (P1) and (P2) are preserved.

Case 6. $\mathscr{L}_{n-1} \cap W = \phi$, $\mathscr{L}_{n-1} \cap W^* = \phi$. Let $\{y_1, \ldots, y_N\}$ be the points in $\mathscr{L}_{n+k_0} \cap \{|z| \leqq 2^{-n+1}\}$ arranged by increasing real part. We may assume either $y_1 = 0$ or y_1 maximizes $|z|$, $z \in K \cap W^* \cap \{2^{-(n+k_0)} \leqq |z| \leqq 2^{-n+1}\}$, and we may assume y_N maximizes $|z|$, $z \in K \cap W \cap \{|z| \leqq 2^{-n+1}\}$. Add on the line segments $[2y_1, y_1]$, $[2y_N, y_N]$ and $[y_j, y_{j+1}]$, $1 \leqq j \leqq N-1$. Let I_Q be as in Case 3. Then (4.1) and (4.2) hold and (P1) and (P2) are preserved.

Remark. By the choice of Γ_0 only Case 1 or Case 2 constructions can happen at z_0 and z_1 until stage k_0, $2^{k_0} = A$. Therefore (P1) and (P2) will always hold at z_0, z_1.

To conclude the proof we note that by Cases 1–6 and estimate (4.1) the quantity $l(\Gamma_n) - l(\Gamma_{n-1})$ is bounded by

$$C \sum_{Q \in \mathscr{Q}_n} \beta^2(Q)\, l(Q) + \frac{1}{2} \sum_{Q \in \mathscr{Q}_n} l(I_Q) \,.$$

Summing from $n = 1$ to N we obtain

$$\begin{aligned} l(\Gamma_N) - l(\Gamma_0) &\leqq C \sum_Q \beta^2(Q)\, l(Q) + \frac{1}{2} \sum_{l(Q) \leqq 2^{-N}} l(I_Q) \\ &\leqq C \sum_Q \beta^2(Q)\, l(Q) + \frac{1}{2} l(\Gamma_N) \,, \end{aligned}$$

the final inequality being a consequence of (4.2). Therefore

$$l(\Gamma_N) \leqq 2 l(\Gamma_0) + C \sum_Q \beta^2(Q)\, l(Q) \,,$$

and taking limits we obtain the first part of the theorem. We note that the segments $[2z_0 - z_1, z_0]$, $[z_1, 2z_1 - z_0] \subset \Gamma_0$ are never altered at any stage of the construction. By throwing them away and taking A large enough we could build Γ so that

$$l(\Gamma) \leqq (1 + \delta)\,\text{diameter}(K) + C_\delta \sum_Q \beta_K^2(Q)\, l(Q)\,,$$

but then (1.5) would not hold for $Q = [0, 1]^2$.

To show property (1.5) holds, we must add some line segments to Γ to form a new curve $\tilde{\Gamma}$. Fix a dyadic cube Q with $\beta(2Q) < \varepsilon_0$ and first suppose that there are points $x_0 \in K \cap Q$ and $x_1, x_2 \in K \cap (5Q \backslash 3Q)$ with the angle between $[x_0, x_1]$ and $[x_0, x_2]$ greater than $\pi/2$. Then the construction yields a subcurve of Γ which connects x_1 to x_2 in S_Q. For the other case (where $x_1, x_2 \in K \cap (5Q \backslash 3Q)$ implies the angle between $[x_0, x_1]$ and $[x_0, x_2]$ is less than $\pi/2$), the construction shows there is an arc $I_Q \subset \Gamma \cap 3Q$ such that distance $(I_Q, K) \geqq C\, l(Q)$. Add to Γ a line segment J_Q crossing $5Q$ in S_Q. Then $l(J_Q) \leqq C l(I_Q)$ and since the I_Q are essentially disjoint from each other,

$$\sum_Q l(J_Q) \leqq C' \sum_Q l(I_Q) \leqq C'\, l(\Gamma)\,.$$

Consequently $l(\tilde{\Gamma}) \leqq C\, l(\Gamma)$.

To show that (1.4) holds, apply Corollary 2 to Γ and obtain $\tilde{\tilde{\Gamma}}$ such that $l(\tilde{\tilde{\Gamma}}) \leqq C_0 + C_0 \beta^2(K)$ and such that the conclusions of Corollary 2 hold. Let $\mathbb{C} \backslash \tilde{\tilde{\Gamma}} = \bigcup \Omega_j$, and suppose $x, y \in \tilde{\tilde{\Gamma}}$. Let $I = [x, y]$ and let $I = E \bigcup \bigcup_j I_j$ be a decomposition of I into $E = I \cap \tilde{\tilde{\Gamma}}$ and open intervals I_j which lie in $\Omega_{k(j)}$. Setting $I_j = [x_j, y_j]$, we have $x_j, y_j \in \partial \Omega_{k(j)}$, and consequently there is an arc $\gamma_j \subset \partial \Omega_{k(j)}$ connecting x_j to y_j such that $l(\gamma_j) \leqq C_0 |x_j - y_j|$. Then if $\gamma = E \bigcup \bigcup_j \gamma_j$, γ is connected, $\gamma \subset \tilde{\tilde{\Gamma}}$, and $l(\gamma) \leqq l(E) + \sum_j l(\gamma_j) \leqq l(E) + C_0 \sum_j |x_j - y_j| \leqq C_0 |x - y|$.

§5. Appendix

In this section we show that (1.2) holds when Γ is a C_0 Lipschitz curve. By using a dilation, we may assume that Γ is given by the parametrization $\psi(\theta) = r(\theta) e^{i\theta}$, where $C_0^{-1} \leqq r(\theta) \leqq 1$, and $|r(\theta_1) - r(\theta_2)| \leqq C_0 |\theta_1 - \theta_2|$. Let Γ_n be the polygon obtained from the line segments

$$J_j^n = [\psi(j 2^{-n+1}\pi), \psi(j+1) 2^{-n+1}\pi)], \qquad 0 \leqq j \leqq 2^n\,.$$

Then J_j^n splits into two intervals $J_{2j}^{n+1}, J_{2j+1}^{n+1}$ at stage $n+1$. Define

$$\delta_{n,j} = 2^{-n} \sup_{z \in J_{2j}^{n+1} \cup J_{2j+1}^{n+1}} \text{distance}\,(z, J_j^n)\,.$$

Then by elementary geometry,

$$l(J_{2j}^{n+1}) + l(J_{2j+n}^{n+1}) - l(J_j^n) \geqq C(\delta_{n,j})^2\, 2^{-n}$$

Summing from $n = 1$ to ∞ we obtain

$$c \sum_{n,j} (\delta_{n,j})^2 2^{-n} \leqq l(\Gamma) . \tag{5.1}$$

$$\sum_{k=n}^{\infty} \sum_{J_m^k \subset \theta_j^n} (\delta_{k,m})^2 2^{-k} \leqq C 2^{-n} .$$

Here we define $\Gamma_j^n = \Gamma \cap \{j 2^{-n+1}\pi \leqq \theta \leqq (j+1) 2^{-n+1}\pi\} \equiv \Gamma \cap \theta_j^n$. Our result will follow if we can show that

$$\tilde{\beta}(\Gamma_j^n) = 2^{-n} \sup_{z \in \Gamma_j^n} \text{distance}\,(z, J_j^n)$$

satisfies

$$\sum_{n,j} \tilde{\beta}(\Gamma_j^n)^2 2^{-n} \leqq C_1 , \tag{5.2}$$

for then we may rotate the dyadic grid through $[0, 2\pi]$ to obtain quantities $\tilde{\beta}_\theta(\Gamma_j^n)$ and note that

$$\sum_{l(Q) = 2^{-n-1}} \beta_\Gamma(Q)^2 \, l(Q) \leqq$$

$$C \int_0^{2\pi} \left\{ \sum_j \tilde{\beta}_\theta(\Gamma_j^n)^2 2^{-n} \right\} d\theta .$$

To prove (5.2) notice that

$$\tilde{\beta}(\Gamma_j^n) \leqq C_1 \sum_{k=1}^{\infty} \sup_{J_m^{n+k} \subset \theta_j^n} 2^{-k} \delta_m^{n+k} .$$

Then (5.2) follows from the above inequality, Minkowski's inequality (or Cauchy–Schwarz), and (5.1).

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Extensions of nilpotent blocks*

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1. Introduction

1.1. Let p be a prime number. In [2] Broué and Puig introduce the so-called *nilpotent blocks*: a block b of a finite group H is said to be *nilpotent* if, for any p-subgroup Q of H and any block f of $C_H(Q)$ associated with b, the quotient $N_H(Q,f)/C_H(Q)$ of the stabilizer of (Q,f) in H by the centralizer of Q is a p-group. In [18] Puig completes the study of such blocks giving the structure of their *source algebras*. As the title announces, the main purpose of this paper is to study the structure – i.e. the source algebra, in the sense of [17] and [18] – of any block of a finite group G lying over a nilpotent block b of a normal subgroup H of G.

1.2. Such a purpose has been motivated by the following situation, which occurs in any block a of any finite group G: consider a p-subgroup Q of G and a block f of $C_G(Q)$ associated with a; if Q is "big enough" – precisely, if Q is self-centralizing in P for any Brauer pair (P, e) containing (Q,f) (cf. [1]) – then $Z(Q)$ is a defect group of f; in that case, f is clearly a nilpotent block of $C_G(Q)$ and therefore our results

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determine the structure of the block of $N_G(Q)$ lying over f. The particular case where Q is a defect group of the full block a of G has been already considered by Reynolds in [21], by Külshammer in [10] and by Puig in Proposition 14.6 of [19], where the source algebra of the block of $N_G(Q)$ lying over f is completely determined.

1.3. On the other hand, if $N_G(Q)$ is p-solvable, the situation above could be handled employing Fong's reduction techniques in [8] – suitably extended to discuss algebra structures as described for instance in [13]. However, notice that our results here are more precise even in this case since they guarantee the "uniqueness" of the final reduction, which is important to connect the structures coming from the different pairs (Q, f).

1.4. Let us come back to our basic situation. Let k be an algebraically closed field of characteristic p, $\mathcal{O}$ a complete discrete valuation ring with residue field k, G a finite group, H a normal subgroup of G, b a primitive idempotent of $Z\mathcal{O}H$, N the stabilizer of b in G, P a maximal p-subgroup of N such that $\mathrm{Br}_P(b) \neq 0$ in $ZkC_H(P)$ and Q the intersection of P and H. Our main results state that, if the block b of H is nilpotent, the structure of any block of G lying over b is completely determined by the following two invariants

1.4.1. a full $\mathcal{O}$-matrix interior P-algebra S which has a P-stable $\mathcal{O}$-basis containing the unity (i.e. a capped endo-permutation $\mathcal{O}P$-module, in Dade's terms cf. [6], S being its $\mathcal{O}$-endomorphism ring).

1.4.2. a twisted group algebra $\mathcal{O}_*\hat{L}$ constructed from a group extension L of N/H by Q, which contains P as a Sylow p-subgroup.

Actually, Theorem 1.8 below shows that the uniqueness of L is stronger than the uniqueness expected from a group extension of N/H by Q: the effort to obtain that uniqueness – which is very important for forthcoming applications – lengthens particularly this paper. Let us mention an easy case: when Q has an N-stable complement K in H – which is the situation considered by Dade in [7] – L is just the quotient N/K.

1.5. To state precisely these results, we will freely use notation and terminology introduced in Sect. 2. Set $A = \mathcal{O}G$, $B = \mathcal{O}H$, $\beta = \{b\}$ so that β is a *point* of H on B and $N = N_G(H_\beta)$ (cf. 2.7). For any $X \subset N$ denote by $\bar{X}$ the image of X in N/H; in particular, $\bar{N} = N/H$ and we consider N, endowed with the action on the set $\bar{N}$ given by left multiplication, as an *$\bar{N}$-permutation group* (cf. 2.4). We consider A endowed with B as a *G-algebra extension* – an interior G-algebra endowed with a G-stable subalgebra containing the unity (cf. 2.9) – but, as we explain below, we are mostly interested in the restriction $\mathrm{Res}_N^G(A)$ (cf. 2.9). Notice that β is still a point of N (and of any subgroup of N containing H) on B and that $H_\beta \subset N_\beta$ (cf. 2.7).

1.6. Let α be the unique point of G on B such that $H_\beta \subset N_\beta \subset G_\alpha$ (i.e. $\alpha = \{\mathrm{Tr}_N^G(b)\}$), γ a local point of P on B such that $P_\gamma \subset N_\beta \subset G_\alpha$ and δ a local point of Q on B such that $Q_\delta \subset P_\gamma$. Notice that γ and δ exist since $\mathrm{Br}_P^B(b) \neq 0$ and therefore $\mathrm{Br}_Q^B(b) \neq 0$ (cf. 2.6 and [1], §1). Moreover, P_γ and Q_δ are respectively *defect pointed*

groups of G_α and H_β since our maximal choice of P guarantees that P_γ is a maximal local pointed group on B contained in N_β, so that $\alpha \subset \mathrm{Tr}_P^G(B^P \cdot \gamma \cdot B^P)$ (cf. 2.7), and then Q_δ is a maximal local pointed group on B contained in H_β (see Proposition 5.3 below for more detail). Consequently, we get $N = H \cdot N_G(Q_\delta)$ by the Frattini argument (cf. [19], 2.10.2) (we have $N_G(Q_\delta) \subset N$ since β is the unique point of H on B such that $Q_\delta \subset H_\beta$); in particular, if the block b of H is nilpotent, the inclusion $N_G(Q_\delta) \subset N$ induces

$$\bar{N} \cong N_G(Q_\delta)/Q \cdot C_H(Q) \cong E_{N,\bar{N}}(Q_\delta) \tag{1.6.1}$$

since $N_H(Q_\delta) = Q \cdot C_H(Q)$ (cf. [18], 1.7.2) and $C_N(Q_\delta, \bar{N}) = C_H(Q)$ (cf. 2.7). A general remark: if $H = Q$ then all the points above coincide with $\{1\}$ and we will omit to mention them.

1.7. We denote respectively by $\tilde{f}_\alpha : A_\alpha \to A$, $\tilde{f}_\beta : A_\beta \to \mathrm{Res}_N^G(A)$ and $\tilde{f}_\varepsilon : A_\varepsilon \to \mathrm{Res}_R^G(A)$ *embedded algebra extensions* associated with G_α, N_β and any local pointed group R_ε on B such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$ (cf. 2.11). Notice that, as $bxb = 0$ for any $x \in G - N$, the embedding $\tilde{f}_\beta$ induces an N-algebra extension isomorphism $A_\beta \cong \mathcal{O}Nb$ where $\mathcal{O}Nb$ is endowed with $\mathcal{O}Hb$ (cf. 2.9), and therefore a G-algebra extension isomorphism (cf. 2.12.2)

$$A_\alpha \cong \mathrm{Ind}_N^G(\mathcal{O}Nb) \tag{1.7.1}$$

which shows that, for our purposes, it suffices to consider A_β, and *a fortiori* $\mathrm{Res}_N^G(A)$. Actually, as we explain from 1.16 on, it suffices to study the P-algebra extension A_γ and Theorem 1.12 below describes its structure, which depends on a suitable group extension L of N/H by Q as we said in 1.4. We first give our main result about L.

Theorem 1.8. *With the notation above, assume that the block b of H is nilpotent. There are a finite group L and group homomorphisms $\pi : L \to \bar{N}$ and $\tau : P \to L$ fulfilling the following conditions:*

1.8.1. We have $\mathrm{Im}(\pi) = \bar{N}$, $\mathrm{Ker}(\pi) = \tau(Q)$, $\mathrm{Ker}(\tau) = \{1\}$ *and* $\pi(\tau(u)) = \bar{u}$ *for any $u \in P$. In particular, considering L endowed with the action on the set $\bar{N}$ given by π and left multiplication as an $\bar{N}$-permutation group, τ induces a bijection* $\tau_R : \tilde{\mathrm{Hom}}_{\bar{N}}(R, P) \cong \tilde{\mathrm{Hom}}_{\bar{N}}(\tau(R), \tau(P))$ *for any subgroup R of P.*

1.8.2. For any local pointed group R_ε on B such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$ we have $\tau_R(E_{N,\bar{N}}(R_\varepsilon, P_\gamma)) = E_{L,\bar{N}}(\tau(R), \tau(P))$.

Moreover, if L' is a finite group and $\pi' : L' \to \bar{N}$ and $\tau' : P \to L'$ are group homomorphisms fulfilling conditions 1.8.1 *and* 1.8.2, *there is a unique group exomorphism $\tilde{\sigma} : L \cong L'$ having a representative σ such that $\pi' \circ \sigma = \pi$ and $\sigma \circ \tau = \tau'$.*

Remark 1.9. Notice that $\tau(P)$ is a Sylow p-subgroup of L (cf. 1.8.1, 1.8.2 and 2.16.2). In particular, any group automorphism ω of L such that $\pi \circ \omega = \pi$ and $\omega \circ \tau = \tau$ is an inner automorphism determined by an element of $C_Q(P)$ (cf. Remark 4.12 below), which proves the uniqueness of $\tilde{\sigma}$ above.

Remark 1.10. In Proposition 5.5 below we show that ε is the unique local point of R on B such that $R_\varepsilon \subset P_\gamma$, and therefore if T_ψ is another local pointed group on B

such that $Q_\delta \subset T_\psi \subset P_\gamma$ we have

$$\tau_{T,R}(E_{N,\bar{N}}(T_\psi, R_\varepsilon)) = E_{L,\bar{N}}(\tau(T), \tau(R)) \tag{1.10.1}$$

where $\tau_{T,R}$ is the bijection from $\tilde{\mathrm{H}}\mathrm{om}_{\bar{N}}(T, R)$ onto $\tilde{\mathrm{H}}\mathrm{om}_{\bar{N}}(\tau(T), \tau(R))$ induced by τ.

1.11. From now on, *we assume that the block b of H is nilpotent.* To construct A_γ from the group L above, we have to introduce a suitable central extension of L by $\mathcal{K}^\times$, obtained by restriction from the following central extension of $\bar{N}$ by $\mathcal{K}^\times$. As b is nilpotent, $B_\beta/J(B_\beta)$ is a simple k-algebra (cf. [18], 1.9.1); denote by $s\colon B_\beta \to B_\beta/J(B_\beta)$ the canonical map and, in particular, set $B_\beta/J(B_\beta) = s(B_\beta)$. The natural action of N on $s(B_\beta)$ defines a group homomorphism $N \to \mathrm{Aut}(s(B_\beta))$ and, considering $s(B_\beta)^\times$ as a central extension of $\mathrm{Aut}(s(B_\beta))$ by $\mathcal{K}^\times$, we set (cf. 2.1 and 2.4 below)

$$\hat{N} = \mathrm{Res}_N(s(B_\beta)^\times)\,; \tag{1.11.1}$$

that is, the elements of $\hat{N}$ are the pairs $(s(c), x)$ where $x \in N$ and $c \in B_\beta^\times$ fulfils $s(d^c) = s(d^x)$ for any $d \in B_\beta$. Moreover, there is a unique group homomorphism $H \to \hat{N}$ mapping $x \in H$ upon $(s(x \cdot 1), x)$, this homomorphism is injective and its image is a normal subgroup of $\hat{N}$ intersecting trivially the image of $\mathcal{K}^\times$. So, identifying H with its image in $\hat{N}$, the quotient $\hat{N}/H$ is a central extension of $\bar{N}$ by $\mathcal{K}^\times$ denoted by $\hat{\bar{N}}$. Recall that, if $\hat{L}$ is a central extension of a finite group L by $\mathcal{K}^\times$, we denote by $\mathcal{O}_*\hat{L}$ the corresponding twisted group algebra (cf. 2.1 and 2.4 below) and by $\mathcal{O}_*\hat{L}^0$ the opposite $\mathcal{O}$-algebra.

Theorem 1.12. *With the notation above, assume that the block b of H is nilpotent. Let L be a finite group and $\pi\colon L \to \bar{N}$ and $\tau\colon P \to L$ group homomorphisms fulfilling conditions* 1.8.1 *and* 1.8.2, *set $\hat{L} = \mathrm{Res}_\pi(\hat{\bar{N}})$ and consider $\mathcal{O}_*\hat{L}^0$, endowed with the group homomorphism $P \to (\mathcal{O}_*\hat{L}^0)^\times$ induced by τ and with the image of $\mathcal{O}Q$, as a P-algebra extension. Then there is an $\mathcal{O}$-simple interior P-algebra S, unique up to isomorphism, such that we have an isomorphism*

$$A_\gamma \cong S \underset{\mathcal{O}}{\otimes} \mathcal{O}_*\hat{L}^0 \tag{1.12.1}$$

as P-algebra extensions, and the determinant maps the image of P in $\mathrm{Ind}_{\tau(P)}^L(S^\tau)^\times$, where S^τ is the restriction of S to $\tau(P)$, onto $\{1\}$. Moreover, S has a P-stable $\mathcal{O}$-basis containing the unity as unique P-fixed element.

Remark 1.13. If S' is an $\mathcal{O}$-simple interior P-algebra such that $A_\gamma \cong S' \otimes_\mathcal{O} \mathcal{O}_*\hat{L}^0$, there is a group homomorphism $\theta\colon L \to \mathcal{O}^\times$ such that $S' \cong \mathrm{Res}_\tau(\mathcal{O}_\theta) \otimes_\mathcal{O} S$ where $\mathcal{O}_\theta$ denotes the corresponding interior L-algebra. Indeed, it is clear that $\mathcal{O}_\theta \otimes_\mathcal{O} \mathcal{O}_*\hat{L}^0 \cong \mathcal{O}_*\hat{L}^0$ as P-algebra extensions and, since p divides neither $\mathrm{rank}_\mathcal{O}(S)$ nor $|L\colon \tau(P)|$ (cf. Remark 1.9), we may choose θ such that the determinant maps the image of P in $(\mathcal{O}_\theta - 1 \otimes_\mathcal{O} \mathrm{Ind}_{\tau(P)}^L(S'^\tau))^\times$ onto $\{1\}$.

Remark 1.14. Since 1.12.1 maps B_γ onto $S \otimes_\mathcal{O} \mathcal{O}Q$ and $P \cdot 1$ onto $\{(u \cdot 1) \otimes u\}_{u \in P}$, this isomorphism induces a P-algebra extension isomorphism from $\sum_{u \in P} u \cdot B_\gamma$ onto $S \otimes_\mathcal{O} \mathcal{O}P$. Hence, since $\tilde{f}_\gamma$ induces an embedding from $\sum_{u \in P} u \cdot B_\gamma$ to $\sum_{u \in P} u \cdot B_\gamma = \mathcal{O}(P \cdot H)$, $S \otimes_\mathcal{O} \mathcal{O}P$ becomes a source algebra of b as a block of $P \cdot H$. This shows that b is still a nilpotent block of $P \cdot H$ (cf. [18], Th. 1.6) and therefore

proves already the uniqueness of the P-algebra S and the existence of a suitable P-stable $\mathcal{O}$-basis of S (cf. [18], Th. 1.6). In particular, as $\mathrm{Aut}(A_\gamma)$ stabilizes $\sum_{u\in P} u\cdot B_\gamma$, we have a group isomorphism

$$\tilde{\mathrm{A}}\mathrm{ut}(\mathcal{O}_*\hat{L}^0) \cong \tilde{\mathrm{A}}\mathrm{ut}(S\underset{\mathcal{O}}{\otimes}\mathcal{O}_*\hat{L}^0) \tag{1.14.1}$$

mapping $\tilde{g}\in\tilde{\mathrm{A}}\mathrm{ut}(\mathcal{O}_*\hat{L}^0)$ upon $\widetilde{\mathrm{id}}_S\otimes\tilde{g}$ (cf. [18], Cor. 6.9 and [15], Prop. 2.1).

1.15. When Q is not normal in H, it is useful to know the following alternative definition of $\hat{\bar{N}}$[1], which depends on a theorem of Dade (cf. 2.12.4). The action of $N_G(Q_\delta)$ on the simple k-algebra $B(Q_\delta)$ (cf. 2.7) defines a group homomorphism $N_G(Q_\delta)\to\mathrm{Aut}(B(Q_\delta))$, and then, considering $B(Q_\delta)^\times$ as a central extension of $\mathrm{Aut}(B(Q_\delta))$ by $k^\times$, we claim that (cf. 2.4).

1.15.1. there is a group extension isomorphism

$$\mathrm{Res}_{N_G(Q_\delta)}(\hat{N}) \cong \mathrm{Res}_{N_G(Q_\delta)}(B(Q_\delta)^\times)$$

mapping $(s(vz\cdot 1), vz)$ *upon* $(s_\delta(z), vz)$ *for any* $v\in Q$ *and any* $z\in C_H(Q)$.

This isomorphism provides another definition of $\hat{\bar{N}}$; indeed, we have $\bar{N}\cong N_G(Q_\delta)/Q\cdot C_H(Q)$ (cf. 1.7.2) and, identifying $Q\cdot C_H(Q)$ with its image in $\mathrm{Res}_{N_G(Q_\delta)}(B(Q_\delta)^\times)$ through the injective group homomorphism mapping vz upon $(s_\delta(z), vz)$, 1.15.1 induces a *group extension isomorphism*

$$\hat{\bar{N}} \cong \mathrm{Res}_{N_G(Q_\delta)}(B(Q_\delta)^\times)/Q\cdot C_H(Q)\,. \tag{1.15.2}$$

Set $\mathrm{Res}_{N_G(Q_\delta)}(\hat{N}) = \hat{N}_G(Q_\delta)$; to prove 1.15.1 it suffices to exhibit a group extension homomorphism $\hat{N}_G(Q_\delta)\to B(Q_\delta)^\times$ extending the group homomorphism $C_H(Q)\to B(Q_\delta)^\times$ induced by s_δ. We have already a group extension homomorphism

$$\hat{N}_G(Q_\delta)\to s(B_\beta)^\times \tag{1.15.3}$$

extending the group homomorphism $C_H(Q)\to s(B_\beta)$ induced by s (cf.1.11). Moreover, the canonical homomorphism $k\otimes_{\mathcal{O}} B_\beta\to s(B_\beta)$ determines a *strict covering exomorphism* of interior H-algebras over k (cf. [18], 4.14); indeed, setting also $s(B_\delta) = B_\delta/J(B_\delta)$, it follows from Theorem 1.6 and Example 4.25 in [18] that the homomorphism $k\otimes_{\mathcal{O}} B_\delta\to s(B_\delta)$ determines a strict covering exomorphism of interior Q-algebras over k (since b is a nilpotent block of H), and therefore it follows from Theorem 3.8 and Corollary 4.23 in [18] that the induced homomorphism $\mathrm{Ind}_Q^H(k\otimes_{\mathcal{O}} B_\delta)\to\mathrm{Ind}_Q^H(s(B_\delta))$ determines a strict covering exomorphism of interior H-algebras over k (since H is *locally controlled* by Q on $k\otimes_{\mathcal{O}} B_\delta$, so on $s(B_\delta)$); now the statement follows from Theorem 3.4 in [15] and Proposition 4.17 in [18][2]. Consequently, $s(\delta)$ is contained in a local point $\tilde{\delta}$ of Q on $s(B_\beta)$ (cf. [18], 4.14 and 4.19), and we have an $N_G(Q_\delta)$-algebra isomorphism (cf. [18], 4.14.2)

$$B(Q_\delta) \cong s(B_\beta)(Q_{\tilde{\delta}}) \tag{1.15.4}$$

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[1] This alternative definition is only employed here in 1.20 below to show a consequence of Theorem 1.12

[2] This argument reproduces line by line the first part of the proof of Proposition 7.2 in [18], and the reader may refer himself there for more detail.

and a group extension isomorphism (cf. 2.12.4, [18], Th. 1.6 and [19], 6.22)

$$\hat{F}_{s(B_\beta)}(Q_{\bar{\delta}}) = \hat{F}_{s(B_\delta)}(Q_{\bar{\delta}}) \cong k^\times \times F_{s(B_\delta)}(Q_{\bar{\delta}}) . \tag{1.15.5}$$

Now it suffices to compose the group extension homomorphism 1.15.3, whose image is contained in $N_{s(B_\beta)^\times}(\ddot{Q}_{\bar{\delta}})$ where $\ddot{Q}$ is the image of Q in $s(B_\beta)^\times$, with the group extension homomorphism $N_{s(B_\beta)^\times}(\ddot{Q}_{\bar{\delta}}) \to B(Q_\delta)^\times$ obtained from 1.15.5, 2.12.3 and 1.15.4.

1.16. We will explain now why Theorem 1.12 provides complete information on the blocks of G lying over b, as announced in 1.7. First of all, by 2.14.1 there is a unique *interior G-algebra* embedding

$$\tilde{g}_\alpha^\gamma \colon A_\alpha \to \mathrm{Ind}_P^G(A_\gamma) \tag{1.16.1}$$

such that $\mathrm{Res}_P^G(\tilde{g}_\alpha^\gamma) \circ \tilde{f}_\gamma^\alpha = \tilde{d}_P^G(A_\gamma)$, and therefore the local pointed groups on A_α are just the G-conjugates of local pointed groups on A_γ (cf. [17], 1.12). Moreover, as $A_\gamma \cong S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L}^0$ as interior P-algebras (cf.1.12.1), for any subgroup R of P there is a bijection $\mathscr{LP}_{A_\gamma}(R) \cong \mathscr{LP}_{\mathcal{O}_*\hat{L}^0}(R)$ mapping $\varepsilon^0 \in \mathscr{LP}_{A_\gamma}(R)$ on the unique $\tau(\varepsilon^0) \in \mathscr{LP}_{\mathcal{O}_*\hat{L}^0}(R)$ such that $m^{\varepsilon^0}_{1 \otimes \tau(\varepsilon^0)} \neq 0$ (cf. 2.3 applied to $1 \otimes (\mathcal{O}_*\hat{L}^0)^R \subset (A_\gamma)^R$, and [18], Th. 5.3), and considering $\mathcal{O}_*\hat{L}^0$ as an L-algebra we set

$$\tau(R)_{\tau(\varepsilon^0)} = \tau(R_{\varepsilon^0}) . \tag{1.16.2}$$

So the local categories of A_α and $\mathcal{O}_*\hat{L}^0$ (cf. 2.8), and the *local fusion category* of A_γ (cf. 2.12) have, up to isomorphism, the same objects and the following statement shows that they have also the same morphisms.

1.16.3. If R_{ε^0} and T_{ψ^0} are local pointed groups on A_γ then $E_G(T_{\psi^0}, R_{\varepsilon^0}) = F_{A_\gamma}(T_{\psi^0}, R_{\varepsilon^0})$ and $\tau_{T,R}(F_{A_\gamma}(T_{\psi^0}, R_{\varepsilon^0})) = E_L(\tau(T_{\psi^0}), \tau(R_{\varepsilon^0}))$ where $\tau_{T,R}$ is the bijection from $\tilde{\mathrm{H}}\mathrm{om}(T, R)$ onto $\tilde{\mathrm{H}}\mathrm{om}(\tau(T), \tau(R))$ induced by τ.

Indeed, by Theorem 3.1 and Proposition 2.14 in [17] we know already that

$$E_G(T_{\psi^0}, R_{\varepsilon^0}) = F_A(T_{\psi^0}, R_{\varepsilon^0}) = F_{A_\gamma}(T_{\psi^0}, R_{\varepsilon^0})$$

and identifying $\tau(P)$ with its unique lifting to $\hat{L}^0$, that

$$E_L(\tau(T_{\psi^0}), \tau(R_{\varepsilon^0})) = F_{\mathcal{O}_*\hat{L}^0}(\tau(T_{\psi^0}), \tau(R_{\varepsilon^0}))$$

(since, by Lemma 5.5 and Proposition 5.15 in [19], $\mathcal{O}_*\hat{L}^0$ can be embedded in a group algebra $\mathcal{O}L'$ of a *finite* central p'-extension L' of L). Moreover, by 1.12.1, $\mathcal{O}_*\hat{L}^0$ can be embedded in $S^0 \otimes_{\mathcal{O}} A_\gamma$ (since $\mathcal{O}$ can be embedded in $S^0 \otimes_{\mathcal{O}} S$) and therefore, again by Proposition 2.14 in [17], we have

$$F_{\mathcal{O}_*\hat{L}^0}(\tau(T_{\psi^0}), \tau(R_{\varepsilon^0})) = F_{S^0 \underset{\mathcal{O}}{\otimes} A_\gamma}(\tau(T_{\psi^0}), \tau(R_{\varepsilon^0})) .$$

Now, 1.16.3 follows from the following general result.

Lemma 1.17[3]. *With the notation above, let D be an interior P-algebra having an $\mathcal{O}$-basis X such that $P \cdot X \cdot P = X$ and $|P \cdot x| = |P| = |x \cdot P|$ for any $x \in X$. If R_ε and T_ψ*

[3] This lemma is not employed in the sequel. Actually, it should have been already in [18], §5!

are local pointed groups on D then

$$F_{S\underset{\mathcal{O}}{\otimes}D}(R_{\varepsilon'}, T_{\psi'}) = F_D(R_\varepsilon, T_\psi) \cap F_S(R_\rho, T_\sigma) \tag{1.17.1}$$

where ρ and σ are respectively the unique local points of R and T on S and ε' and ψ' the unique local points of R and T on $S \otimes_{\mathcal{O}} D$ such that

$$\mathrm{Br}_R^S(\rho) \otimes \mathrm{Br}_R^D(\varepsilon) \subset \mathrm{Br}_R^{S\otimes D}(\varepsilon') \qquad \textit{and} \qquad \mathrm{Br}_T^S(\sigma) \otimes \mathrm{Br}_T^D(\psi) \subset \mathrm{Br}_T^{S\otimes D}(\psi')$$

Proof. It suffices to prove that $F_S(R_\rho, T_\sigma)$ contains $F_{S\otimes_{\mathcal{O}} D}(R_{\varepsilon'}, T_{\psi'})$ (cf. [18], Th. 5.3.). Moreover, if $R_{\varepsilon'} \subset T_{\psi'}$ then $R \subset T$ and therefore $R_\rho \subset T_\sigma$; hence, we may assume that $|R| = |T|$ (cf. [17], 2.11).

The existence of X shows that the canonical image of $\mathcal{O}R$ in D_ε is isomorphic to $\mathcal{O}R$ and a direct summand of D_ε as $\mathcal{O}(R \times R)$-modules by left and right multiplication (cf. the proof of 3.4 in [17]); hence $\mathcal{O}R$ is a direct summand of $S \underset{\mathcal{O}}{\otimes} D_\varepsilon$, so a direct summand of $(S \underset{\mathcal{O}}{\otimes} D_\varepsilon)_{\varepsilon'}$, as $\mathcal{O}(R \times R)$ modules since

$$\mathrm{Res}^{P\times P}_{R\times R}(S) \underset{\mathcal{O}}{\otimes} \mathcal{O}R \cong \mathrm{Res}^{P\times P}_{R\times R}(S) \underset{\mathcal{O}}{\otimes} \mathrm{Ind}^{R\times R}_{\delta(R)}(\mathcal{O}) \cong \mathrm{Ind}^{R\times R}_{\delta(R)}(\mathrm{Res}^{P\times P}_{\delta(R)}(S))$$

where $\delta(R)$ is the diagonal subgroup of $R \times R$. In particular, if $j' \in \varepsilon'$ there are $b' \in j'(S \otimes_{\mathcal{O}} D_\varepsilon)^R j'$ and an $\mathcal{O}(R \times R)$-submodule M of $j'(S \underset{\mathcal{O}}{\otimes} D_\varepsilon)\, j'$ such that

$$j'(S \underset{\mathcal{O}}{\otimes} D) j' \cong M \oplus \left(\bigoplus_{v \in R} \mathcal{O}(v \cdot b') \right).$$

Let $\tilde{\varphi}$ be an element of $F_{S\otimes_{\mathcal{O}} D}(R_{\varepsilon'}, T_{\psi'})$, denote by $\delta_\varphi\colon R \to R \times T$ the homomorphism mapping $v \in R$ upon $(v, \varphi(v))$ and choose $a' \in (S \otimes_{\mathcal{O}} D)^\times$ such that $j'^{a'} \in \psi'$ and $(v \cdot j')^{a'} = j'^{a'} \cdot \varphi(v)$ for any $v \in R$ (cf. [17], Prop. 2.12). It is now clear that $\mathrm{Ind}^{R\times T}_{\delta_\varphi(R)}(\mathcal{O})$ is a direct summand of $j'(S \otimes_{\mathcal{O}} D_\varepsilon)\, j'a'$, so a direct summand of $S \otimes_{\mathcal{O}} D$, as $\mathcal{O}(R \times T)$-modules; hence, $S^0 \otimes_{\mathcal{O}} \mathrm{Ind}^{R\times T}_{\delta_\varphi(R)}(\mathcal{O})$ is a direct summand of $S^0 \otimes_{\mathcal{O}} S \otimes_{\mathcal{O}} D$ always as $\mathcal{O}(R \times T)$-modules. Consequently, since $S \otimes_{\mathcal{O}} S^0 \cong \mathrm{End}_{\mathcal{O}}(S)$ as interior P-algebras and

$$S^0 \underset{\mathcal{O}}{\otimes} \mathrm{Ind}^{R\times T}_{\delta_\varphi(R)}(\mathcal{O}) \cong \mathrm{Ind}^{R\times T}_{\delta_\varphi(R)}(\mathrm{Res}^{P\times P}_{\delta_\varphi(R)}(S^0))$$

as $\mathcal{O}(R \times T)$-modules, on one hand $S \otimes_{\mathcal{O}} S^0 \otimes_{\mathcal{O}} D$ is a permutation $\mathcal{O}(P \times P)$-module and therefore $\mathrm{Res}^{P\times P}_{\delta_\varphi(R)}(S^0)$ is a permutation $\mathcal{O}R$-module; on the other hand we have

$$\mathrm{Res}_\varphi(\mathrm{Res}^P_T(S))^0 \underset{\mathcal{O}}{\otimes} \mathrm{Res}^P_R(S) \cong \mathrm{End}_{\mathcal{O}}(\mathrm{Res}^{P\times P}_{\delta_\varphi(R)}(S^0))$$

as interior R-algebras (we identify R with $\delta_\varphi(R)$).

Finally, since $\mathcal{O}$ can be embedded in $\mathrm{End}_{\mathcal{O}}(\mathrm{Res}^{P\times P}_{\delta_\varphi(R)}(S^0))$ and in $\mathrm{End}_{\mathcal{O}}(\mathrm{Res}^{P\times P}_{\delta(R)}(S^0))$, it follows that S_ρ and $\mathrm{Res}_\varphi(S_\sigma)$ can be embedded in $\mathrm{End}_{\mathcal{O}}(\mathrm{Res}^{P\times P}_{\delta_\varphi(R)}(S^0)) \otimes_{\mathcal{O}} \mathrm{Res}^P_R(S)$ and therefore $S_\rho \cong \mathrm{Res}_\varphi(S_\sigma)$ since R has a unique local point on that interior R-algebra; hence $\tilde{\varphi} \in F_S(R_\rho, T_\sigma)$ (cf. [17], 2.18).

1.18. Another consequence of the isomorphism 1.12.1, together with 1.16.1, is that the $\mathcal{O}$-algebras A_α, A_γ and $\mathcal{O}_* \hat{L}^0$ are pairwise *Morita equivalent*; in particular, we get

evident isomorphisms

$$Z(A_\alpha) \cong Z(A_\gamma) \cong Z(\mathcal{O}_*\hat{L}^0) \tag{1.18.1}$$

which induce bijections between the sets of blocks of these $\mathcal{O}$-algebras. Let a^0 be a block of G lying over b, so that $\alpha^0 = \{a^0\}$ is a point of G on A_α, and denote by a_γ^0 the corresponding primitive idempotent of $Z(A_\gamma)$ and by $\tau(\alpha^0) = \{\tau(a^0)\}$ the corresponding point of L on $\mathcal{O}_*\hat{L}^0$; notice that $A_\gamma a_\gamma^0$ is still an interior P-algebra and that we have

$$A_\gamma a_\gamma^0 \cong S \underset{\mathcal{O}}{\otimes} (\mathcal{O}_*\hat{L}^0)\tau(a^0) \tag{1.18.2}$$

as interior P-algebras. Now it is quite clear from 1.16.3 that, for any local pointed group R_{ε^0} on A_γ, the following conditions are equivalent

1.18.3. R_{ε^0} *is a defect pointed group of* G_{α^0}

1.18.4. R_{ε^0} *is a maximal local pointed group on* $A_\gamma a_\gamma^0$

1.18.5. $\tau(R_{\varepsilon^0})$ *is a defect pointed group of* $L_{\tau(a^0)}$

and, in that case, an embedded algebra associated with the unique local point of R on $\mathrm{Res}_R^P(S) \otimes_{\mathcal{O}} (\mathcal{O}_*\hat{L}^0)_{\tau(\varepsilon^0)}$ (cf. [19], 2.13) and [18], Th. 5.3) is a source algebra of a^0: in particular, as we say above (cf. 1.1), all the invariants currently associated with a^0 can be computed from $\mathrm{Res}_R^P(S) \otimes_{\mathcal{O}} (\mathcal{O}_*\hat{L}^0)_{\tau(\varepsilon^0)}$ and therefore related to the corresponding invariants of $\tau(a^0)$ (recall that, by Lemma 5.5 and Proposition 5.15 in [19], $\mathcal{O}_*\hat{L}^0$ can be embedded in the group algebra of a *finite* central p'-extension L' of L, and therefore $\tau(a^0)$ is a block of L' too).

1.19. For instance, we will describe the relationship between the full matrices of *generalized decomposition numbers of* a^0 and $\tau(a^0)$. Assume that $\mathrm{Char}(\mathcal{O}) = 0$ and denote by $\mathcal{K}$ the quotient field of $\mathcal{O}$, by $\mathrm{Irr}_{\mathcal{K}}(G, a^0)$ the set of irreducible $\mathcal{K}$-characters of G associated with a^0 and, for any $\chi \in \mathrm{Irr}_{\mathcal{K}}(G, a^0)$, by χ^τ the corresponding irreducible $\mathcal{K}$-character of $\hat{L}^0$ (recall that 1.12.1 and 1.16.1 induce a Morita equivalence between Aa^0 and $(\mathcal{O}_*\hat{L}^0)\tau(a^0)$). Let R_{ε^0} be a defect pointed group of G_{α^0} such that $R \subset P$ and $m_{\varepsilon^0}^\gamma \neq 0$ (cf. 1.16), and $\mathcal{U}_{\varepsilon^0}$ a set of representatives $u_{\psi^0} \in R_{\varepsilon^0}$ for the G-conjugacy classes of local pointed elements on Aa^0. Notice that, with evident notation, it follows from 1.16.3 that $\{\tau(u_{\psi^0})\}_{u_{\psi^0} \in \mathcal{U}_{\varepsilon^0}}$ is a set of representatives for the L-conjugacy classes of local pointed elements on $(\mathcal{O}_*\hat{L}^0)\tau(a^0)$. If $\chi \in \mathrm{Irr}_{\mathcal{K}}(G, a^0)$ and $u_{\psi^0} \in \mathcal{U}_{\varepsilon^0}$, we denote by $\chi(u_{\psi^0})$ and $\chi^\tau(\tau(u_{\psi^0}))$ the corresponding generalized decomposition numbers; moreover, for any $u \in P$, we denote by $\omega(u)$ the "sign" of the trace of the image of u in $S^\times$ (cf. [18], 1.11.1). Now, as in 1.12 of [18], it is not difficult to prove that:

1.19.1. For any $\chi \in \mathrm{Irr}_{\mathcal{K}}(G, a^0)$ *and any* $u_{\psi^0} \in \mathcal{U}_{\varepsilon^0}$ *we have*

$$\chi(u_{\psi^0}) = \omega(u)\chi^\tau(\tau(u_{\psi^0}))\,.$$

1.20. A last remark. Setting $G' = N_G(Q_\delta)$, $H' = Q \cdot C_H(Q)$, $A' = \mathcal{O}G'$, $B' = \mathcal{O}H'$ and denoting by b' the block of H' such that $\mathrm{Br}_Q^B(b')\mathrm{Br}_Q^B(\delta) = \mathrm{Br}_Q^B(\delta)$ (recall that $B(Q) \cong \mathcal{O}C_H(Q)$), it is clear that b' has a *nilpotent local structure* (cf. [18], 1.7). Then, with evident notation (cf. 1.6 and 1.7), we have $\alpha' = \beta' = \{b'\}$, $N' = G'$,

$\bar{N}' \cong \bar{N}$ (cf. 1.7.2) and, for any local pointed group R_ε on B such that $Q_\delta \subset R_\varepsilon$, there is a unique local pointed group $R_{\varepsilon'}$ on B' such that $\mathrm{Br}_Q^B(\varepsilon') = \mathrm{Br}_Q^B(\varepsilon)$ (since $\mathrm{Br}_Q^B(B'^R) \cong \mathcal{k}C_H(Q)^R$ and $B(R) \cong \mathcal{k}C_H(R)$); moreover, it is clear that $R_{\varepsilon'} \subset P_{\gamma'}$ if and only if $R_\varepsilon \subset P_\gamma$ (since both inclusions are equivalent to $R_{\mathrm{Br}_Q^B(\varepsilon)} \subset P_{\mathrm{Br}_Q^B(\gamma)}$). On the other hand, it follows from 1.8.1 that $Q = H \cap P$ and from 1.13.1 that δ is the unique local point of Q on B such that $Q_\delta \subset P_\gamma$ (cf. [18], th. 5.3); in particular, if R_ε is a local pointed group on B such that $Q_\delta \subset R_\varepsilon$ and x an element of N such that $(R_\varepsilon)^x \subset P_\gamma$, x normalizes Q_δ. Consequently, identifying $\bar{N}'$ with $\bar{N}$ through 1.7.2 and thus considering N' as an $\bar{N}$-*permutation group* (since $N' \subset N$), we have

$$E_{N,\bar{N}}(R_\varepsilon, P_\gamma) = E_{N',\bar{N}}(R_{\varepsilon'}, P_{\gamma'}) \tag{1.20.1}$$

for any local pointed group R_ε on B such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$, and therefore

1.20.2. the group L and the group homomorphisms π: $L \to \bar{N}$ and τ: $P \to L$ fulfil conditions 1.8.1 and 1.8.2 with respect to G', H' and b'.

That is, the group $N_G(Q_\delta)$ and the block $f = b'$ determine L and, since $B(Q_\delta) \cong B'(Q_{\delta'})$ as $N_G(Q_\delta)$-algebras, they determine the central $\mathcal{k}^\times$-extension $\hat{L}$ and the interior P-algebra $\mathcal{O}_*\hat{L}^0$ too (cf. 1.15.2); this shows, for instance, that (cf. 1.12.1 and 1.16.1).

1.20.3. the $\mathcal{O}$-algebras $\mathcal{O}Gb$ and $\mathcal{O}N_G(Q_\delta)f$ are both isomorphic to full matrix algebras over $\mathcal{O}_*\hat{L}^0$

which generalizes the main result of [7].

This paper is divided in eight sections. In §2 we list notation, terminology and quoted results; although we follow essentially the point of view introduced in [15], it is handy to replace *interior G-algebras* by slightly more general objects named *G-algebra extensions* – which not only simplify statements but suggest the adequate equivalence classes of homomorphisms that we may consider – and in §2 we develop the corresponding background. In Sect. 3 we study the so-called *semi-covering exomorphisms*: this class of G-algebra exomorphisms has been introduced by Puig some time ago to understand the relationship between $\mathcal{O}G$ and $\mathcal{O}(G/Z)$ when Z is a central p-subgroup of G[4]; it turns out that the same class is useful to study the relationship between $\mathcal{O}H$ and $\mathcal{O}G$ when H is a normal subgroup of G such that G/H is a p-group, namely to study *p-extensions of blocks*; actually, together with the main result of [18], it is the most important tool to prove Theorem 1.12 when G/H is a p-group.

It is not difficult to prove that $P/Z(Q)$ is a Sylow p-subgroup of $N_G(Q_\delta)/C_H(Q)$ and that the element of $\mathbb{H}^2(P/Z(Q), Z(Q))$ determined by P belongs to the image of $\mathbb{H}^2(N_G(Q_\delta)/C_H(Q), Z(Q))$ and therefore determines a group extension L of $\bar{N}$ by Q (cf. 1.7.2); but such an approach has the disadvantage that L is not necessarily unique up to a *unique* group exoisomorphism; an important part of this paper is devoted to avoid this difficulty, and in Sect. 4 we expose the general criteria we need

[4] In order to prove the existence of a bound for the dimension of a block source algebra in terms of Cartan integers and dimensions of the irreducible source modules. This result has been announced in May 1982, in an Oberwolfach meeting, and will appear in a forthcoming paper

to get *unique* group exomorphisms between group extensions. In Sect. 5 we discuss to what extent the *local category* of B_α (cf. 2.8) is determined by $\bar{N}$. In §6 we essentially handle Theorem 1.12 when $\bar{N}$ is a p-group. In Sect. 7 we exhibit a wide class of conditions which answer the question on the uniqueness of L raised above: we have chosen conditions 1.8.1 and 1.8.2 as the easiest ones to state. Finally, in Sect. 8 we prove Theorems 1.8 and 1.12.

2. Notation, terminology and quoted results

2.1. We follow essentially the notation and terminology introduced in [15], [17], [18] and [19] except that we often replace groups by *permutation groups* and interior G-algebras by *G-algebra extensions* (both terms are defined below). Throughout the paper p is a prime number, k an algebraically closed field of characteristic p and $\mathcal{O}$ a complete discrete valuation ring with residue field k (we allow the case $\mathcal{O} = k$). For any $\lambda \in \mathcal{O}$ we denote by $\bar{\lambda}$ the image of λ in k; more generally, if M is a torsion free $\mathcal{O}$-module, we denote by $\bar{X}$ the image of any $X \subset M$ in $k \otimes_{\mathcal{O}} M$. Recall that the canonical group homomorphism $\mathcal{O}^\times \to k^\times$ has a unique section (cf. [22], II, Prop. 8); we denote by $\check{\lambda}$ the canonical lifting of any $\lambda \in k^\times$ to $\mathcal{O}^\times$.

2.2. All the $\mathcal{O}$-algebras we consider are associative with unity and $\mathcal{O}$-free of finite rank as $\mathcal{O}$-modules. An $\mathcal{O}$-algebra isomorphic to a finite direct product of full matrix algebras over $\mathcal{O}$ is shortly called *$\mathcal{O}$-semisimple*, and *$\mathcal{O}$-simple* if there is just one factor. Let A be an $\mathcal{O}$-algebra: all the A-modules we consider are $\mathcal{O}$-free of finite rank; we denote by A^0 the opposite $\mathcal{O}$-algebra, by $A^\times$ the group of invertible elements of A, by $Z(A)$ the center, by $\mathrm{Aut}(A)$ the group of automorphisms, by $J(A)$ the Jacobson radical and by $\mathcal{P}(A)$ the set of $A^\times$-conjugacy classes of primitive idempotents – shortly called *points* – of A. Whereas, if S is a set, $\mathbb{S}(S)$ denotes the set of all subsets of S to avoid confusion. An ideal $\mathfrak{a}$ of A means always a *two-sided* ideal, and we set $J(\mathfrak{a}) = \mathfrak{a} \cap J(A)$ and denote by $\mathcal{P}(\mathfrak{a})$ the set of $\alpha \in \mathcal{P}(A)$ such that $\alpha \subset \mathfrak{a}$. For any $\alpha \in \mathcal{P}(A)$ we denote by $A(\alpha)$ the simple quotient of A associated with α, by $s_\alpha\colon A \to A(\alpha)$ the canonical homomorphism, by $A \cdot \alpha \cdot A$ the ideal of A generated by α, and by m_α the dimension of any simple $A(\alpha)$-module.

2.3. A homomorphism $f\colon A \to A'$ between $\mathcal{O}$-algebras is not required to be unitary. For any $\alpha \in \mathcal{P}(A)$ and any $\alpha' \in \mathcal{P}(A')$ we denote by $m(f)^\alpha_{\alpha'}$ (or $m^\alpha_{\alpha'}$ when no confusion is possible) the dimension of $s_{\alpha'}(f(i)) \cdot M'$ where $i \in \alpha$ and M' is a simple $A'(\alpha')$-module. In non-commutative algebra it is handy to consider homomorphisms up to inner automorphisms: an *exomorphism* $\tilde{f}$ from A to A' is the orbit of a homomorphism $f\colon A \to A'$ in $\mathrm{Hom}(A, A')$ under the evident action of the group $A^\times \times A'^\times$. Notice that orbits of $A^\times \times A'^\times$ and $A'^\times$ on $\mathrm{Hom}(A, A')$ coincide (cf. [19], 2.3), and therefore

2.3.1. exomorphisms of $\mathcal{O}$-algebras can be composed.

We denote by $\tilde{\mathrm{Hom}}(A, A')$ the set of these orbits. We say that $\tilde{f} \in \tilde{\mathrm{Hom}}(A, A')$ is an *embedding* if $\mathrm{Ker}(f) = \{0\}$ and $\mathrm{Im}(f) = f(1)A'f(1)$.

2.4. If G is a group, we denote by $Z(G)$ the center of G, by $D(G)$ the commutator subgroup of G, by $O_p(G)$ the subgroup generated by the normal finite p-subgroups of G, by $O^p(G)$ the intersection of the normal subgroups of G such that the quotient is a finite p-group, and by $N_G(H)$ and $C_G(H)$ the normalizer and the centralizer of a subgroup H of G. If $\hat{G}$ is an extension of G by an (abelian) group Z and $\psi: H \to G$ is a group homomorphism, we denote by $\mathrm{Res}_\psi(\hat{G})$ the extension of H by Z formed by the group $\hat{G} \underset{G}{\times} H$ of pairs $(\hat{x}, y) \in \hat{G} \times H$ such that $\psi(y)$ is the image of $\hat{x}$ in G and the group homomorphisms $Z \to \hat{G} \underset{G}{\times} H$ mapping $z \in Z$ upon $(z, 1)$ and $\hat{G} \underset{G}{\times} H \to H$ mapping $(\hat{x}, y) \in \hat{G} \underset{G}{\times} H$ upon y; we write $\mathrm{Res}_H(\hat{G})$ instead of $\mathrm{Res}_\psi(\hat{G})$ when the choice of ψ is evident from the context. If $\varphi: Z \to \mathcal{O}^\times$ is a group homomorphism (and G is finite), we may consider the twisted group algebra

$$\mathcal{O}\hat{G} \Big/ \sum_{z \in Z, \hat{x} \in \hat{G}} \mathcal{O} \cdot (z\hat{x} - \varphi(z)\hat{x}) \tag{2.4.1}$$

that we denote by $\mathcal{O}_*\hat{G}$ when there is no confusion on φ, namely if $Z = k^\times$ and φ is the canonical section (cf. 2.1). But in this paper we will systematically consider permutation groups rather than groups: if S is a set, an *S-permutation group* G is a group – still denoted by G – endowed with an action on S (i.e. with a group homomorphism from G to the symmetric group $\Sigma(S)$ of S); then any subgroup of G is considered as an S-permutation group with respect to the restricted action. Notice that, as any group can be considered as an $\varnothing$-permutation group, everything about permutation groups specializes to groups (and then we will omit to mention the empty set).

2.5. Let G and G' be respectively S- and S'-permutation groups for some sets S and S'; a *homomorphism* from G to G' (or from (G, S) to (G', S') to avoid confusion) is a pair (φ, f) where $\varphi: G \to G'$ is a group homomorphism and $f: S \to S'$ a map such that $f(x \cdot s) = \varphi(x) \cdot f(s)$ for any $x \in G$ and any $s \in S$. If $S' = S$, we denote by $\mathrm{Hom}_S(G, G')$ the set of *S-permutation group homomorphisms* from G to G'; similarly, we denote by $\mathrm{Aut}_S(G)$ and $\mathrm{Aut}_S(G')$ the groups of S-permutation group automorphisms of G and G'. The canonical maps $G \to \mathrm{Aut}(G)$ and $G' \to \mathrm{Aut}(G')$, and the structural homomorphisms $G \to \Sigma(S)$ and $G' \to \Sigma(S)$ induce evident group homomorphisms $G \to \mathrm{Aut}_S(G)$ and $G' \to \mathrm{Aut}_S(G')$; then composition induces an action of $G \times G'$ on $\mathrm{Hom}_S(G, G')$ and as above we call *S-permutation group exomorphism* from G to G' any orbit of $G \times G'$ on $\mathrm{Hom}_S(G, G')$ and denote by $\tilde{\mathrm{H}}\mathrm{om}_S(G, G')$ the set of these orbits. Again $G \times G'$ and G' have the same orbits on $\mathrm{Hom}_S(G, G')$, and therefore

2.5.1. exomorphisms of S-permutation groups can be composed.

Moreover we have an evident map from $\tilde{\mathrm{H}}\mathrm{om}_S(G, G')$ to $\tilde{\mathrm{H}}\mathrm{om}(G, G')$ $(= \tilde{\mathrm{H}}\mathrm{om}_\varnothing(G, G'))$.

2.6. Let S be a set and G a finite S-permutation group. *A G-algebra* B (over $\mathcal{O}$) is an $\mathcal{O}$-algebra endowed with a group homomorphism $\varphi: G \to \mathrm{Aut}(B)$; we usually write b^x instead of $\varphi(x^{-1})(b)$ for any $x \in G$ and any $b \in B$. If H is a subgroup of G, B^H denotes the unitary subalgebra of H-fixed elements of B and, for any subgroup

K of H, $\mathrm{Tr}_K^H: B^K \to B^H$ denotes the relative trace map and B_K^H its image (cf. [19], 2.8). For any p-subgroup P of G we set $B(P) = B^P/(\sum_Q B_Q^P + J(\mathcal{O})\cdot B^P)$ where Q runs over the set of proper subgroups of P, and we denote by $\mathrm{Br}_P: B^P \to B(P)$ (or Br_P^B to avoid confusion) the canonical homomorphism (see [19], 2.9 for more detail).

2.7. Let B be a G-algebra. A *pointed group* H_β *on* B is a pair formed by a subgroup H of G (considered as an S-permutation group!) and a point β of B^H; we say that β is a *point of* H *on* B, set $B(H_\beta) = B^H(\beta)$ (cf. 2.2) and denote respectively by $N_G(H_\beta)$ and $C_G(H_\beta, S)$ the stabilizers of β in $N_G(H)$ and in the intersection of $C_G(H)$ with the kernel of the action of G on S; if $H = \langle x \rangle$ we say that x_β is a *pointed element* on B. If K_γ is another pointed group on B, we write $K_\gamma \subset H_\beta$ (or $y_\gamma \in H_\beta$ if $K = \langle y \rangle$) and say that K_γ *is contained in* H_β if $K \subset H$ and for any $i \in \beta$ there is $j \in \gamma$ such that $ij = j = ji$. A pointed group P_γ on B (or a point γ of P on B) is *local* if $\mathrm{Br}_P(\gamma) \neq \{0\}$; we denote by $\mathscr{LP}_B(P)$ the set of local points of P on B and coherently we set $\mathscr{P}_B(H) = \mathscr{P}(B^H)$ (cf. 2.2). A *defect pointed group* P_γ of H_β is a maximal local pointed group on B such that $P_\gamma \subset H_\beta$ or, equivalently, a minimal pointed group on B such that $\beta \subset \mathrm{Tr}_P^H(B^P \cdot \gamma \cdot B^P)$ (see [19], 2.10 for more detail). Notice that any subgroup and any element of G can be respectively considered as a pointed group and a pointed element on the trivial G-algebra $\mathcal{O}$ (and then we will omit to mention the point, which is always $\{1\}$).

2.8. Let H_β and K_γ be pointed groups on B. Any element x of G such that $(K_\gamma)^x \subset H_\beta$ (where $(K_\gamma)^x = (K^x)_{\gamma^x}$) determines an S-permutation group homomorphism from K to H mapping $y \in K$ on y^x and $s \in S$ on $x^{-1}\cdot s$ (cf. 2.5); we denote by $(K_\gamma, x, H_\beta)_S$ the corresponding exomorphism and say that $(K_\gamma, x, H_\beta)_S$ is a G-*exomorphism* (or a (G, S)-*exomorphism* to avoid confusion) *from* K_γ *to* H_β; that is, $(K_\gamma, x, H_\beta)_S$ is an element of $\tilde{\mathrm{H}}\mathrm{om}_S(K, H)$ (cf. 2.5) and, if y is another element of G such that $(K_\gamma)^y \subset H_\beta$, we have $(K_\gamma, x, H_\beta)_S = (K_\gamma, y, H_\beta)_S$ if and only if $y = zxh$ for some $z \in C_G(K_\gamma, S)$ and some $h \in H$ (see [12], Ch. I, §1 and [17], Def. 2.1 for similar definitions). We denote by $E_{G,S}(K_\gamma, H_\beta)$ the subset of $\tilde{\mathrm{H}}\mathrm{om}_S(K, H)$ of G-exomorphisms from K_γ to H_β and set $E_{G,S}(H_\beta) = E_{G,S}(H_\beta, H_\beta)$. Notice that the canonical map $\tilde{\mathrm{H}}\mathrm{om}_S(K, H) \to \tilde{\mathrm{H}}\mathrm{om}(K, H)$ (cf. 2.5) induces a surjective map (cf. [17], Def. 2.1)

$$E_{G,S}(K_\gamma, H_\beta) \to E_G(K_\gamma, H_\beta)\ (= E_{G,\varnothing}(K_\gamma, H_\beta))\,. \tag{2.8.1}$$

Moreover, as any composition of G-exomorphisms is a G-exomorphism, $E_{G,S}(H_\beta)$ is a subgroup of $\tilde{\mathrm{A}}\mathrm{ut}_S(H)$ isomorphic to $N_G(H_\beta)/H\cdot C_G(H_\beta, S)$. We call *local category* of B the category where objects are the local pointed groups on B and morphisms are the corresponding G-exomorphisms. If P_γ is a local pointed group on B, we denote by $C_G^P(P_\gamma, S)$ the inverse image of $O_p(E_{G,S}(P_\gamma))$ in $N_G(P_\gamma)$.

2.9. An *interior* G-*algebra* A (over $\mathcal{O}$) is an $\mathcal{O}$-algebra endowed with a group homomorphism $\varphi: G \to A^\times$; we usually write $x\cdot a\cdot y$ instead of $\varphi(x)a\varphi(y)$ (see [19], 2.11 for more detail). But in this paper it is handy to consider a slightly more general structure: a G-*algebra extension* A is an interior G-algebra – still denoted by A – endowed with a G-stable subalgebra B containing the unity of A. Then B has

an evident G-algebra structure; conversely, as in 2.7 of [18], we denote by BG the interior G-algebra formed by the free B-module over G endowed with the evident product fulfilling

$$(bx)(cy) = bc^{x^{-1}}xy$$

for any $x, y \in G$ and $b, c \in B$; except when a different G-stable unitary subalgebra will be explicit, we consider BG endowed with Be, where e is the unit element of G, as a G-algebra extension. If $\psi: H \to G$ is an S-permutation group homomorphism, we denote by $\mathrm{Res}_\psi(A)$ the H-algebra extension defined by the group homomorphism $\varphi \circ \psi: H \to A^\times$ and the same subalgebra B; when H is a subgroup of G and ψ the inclusion map (with id_S), we set $\mathrm{Res}_\psi(A) = \mathrm{Res}^G_H(A)$. Notice that if $B = A$ then we have nothing more than the interior G-algebra A and therefore everything about G-algebra extensions specializes to interior G-algebras. Similarly, if $A = BG$ then the whole structure depends only on the G-algebra B and therefore everything about G-algebra extensions specializes to G-algebras by restricting to the structural subalgebra; in particular, this remark applies to the *exomorphisms* defined below.

2.10. Let A and A' be G-algebra extensions and denote respectively by B and B' the structural subalgebras. We denote by $A \otimes_{\mathcal{O}} A'$ the G-algebra extension formed by the tensor product of A and A' endowed with the subalgebra $B \otimes_{\mathcal{O}} B'$ and the evident diagonal group homomorphism

$$G \to A^\times \otimes (A')^\times \subset (A \underset{\mathcal{O}}{\otimes} A')^\times .$$

A homomorphism f from A to A' is an $\mathcal{O}$-algebra homomorphism such that $f(B) \subset B'$ and $f(x \cdot a \cdot y) = x \cdot f(a) \cdot y$ for any $x, y \in G$ and any $a \in A$. We denote by $\mathrm{Hom}(A, A')$ the set of *G-algebra extension homomorphisms* from A to A'. It is clear that conjugation induces group homomorphisms from $(B^G)^\times$ to $\mathrm{Aut}(A)$ and from $(B'^G)^\times$ to $\mathrm{Aut}(A')$, and then composition induces an action of $(B^G)^\times \times (B'^G)^\times$ on $\mathrm{Hom}(A, A')$: an *exomorphism $\tilde{f}$ from A to A'* is the orbit of a homomorphism $f: A \to A'$ under this action; again this orbit coincides with the orbit of f under the action of $(B'^G)^\times$ and therefore

2.10.1. exomorphisms of G-algebra extensions can be composed.

We denote by $\tilde{\mathrm{H}}\mathrm{om}(A, A')$ the set of these orbits. We say that $\tilde{f} \in \tilde{\mathrm{H}}\mathrm{om}(A, A')$ is an *embedding* if $\mathrm{Ker}(f) = \{0\}$, $f(A) = f(1)A'f(1)$ and $f(B) = f(1)B'f(1)$; notice that

2.10.2. If $\tilde{f}: A \to A'$ is an embedding, for any pointed group H_β on B, $f(\beta)$ is contained in a unique point β' of H on B', we have $f^{-1}(\beta') = \beta$ and $\tilde{f}$ induces a k-algebra embedding $\tilde{f}(H_\beta): B(H_\beta) \to B'(H_{\beta'})$.

Often we denote β and β' by the same letter. If $\psi: H \to G$ is an S-permutation group homomorphism and $\tilde{f} \in \tilde{\mathrm{H}}\mathrm{om}(A, A')$, we denote by $\mathrm{Res}_\psi(\tilde{f})$ the exomorphism from $\mathrm{Res}_\psi(A)$ to $\mathrm{Res}_\psi(A')$ (cf. 2.9) containing $\tilde{f}$; as above, when H is a subgroup of G and ψ the inclusion map (with id_S), we set $\mathrm{Res}_\psi(\tilde{f}) = \mathrm{Res}^G_H(\tilde{f})$; notice that (as in 2.12.2 of [19])

2.10.3. if $\tilde{f}, \tilde{g} \in \tilde{\mathrm{H}}\mathrm{om}(A, A')$ *we have* $\mathrm{Res}_\psi(\tilde{f}) = \mathrm{Res}_\psi(\tilde{g})$ *if and only if* $\tilde{f} = \tilde{g}$.

Indeed, if $b' \in (B')^\times$ fulfils $g(a) = f(a)^{b'}$ for any $a \in A$ then the right multiplication by b' induces a $B'G$-module isomorphism from $B'f(1)$ onto $B'g(1)$ since we have

$$(a')^x b' = x^{-1} \cdot a' f(1 \cdot x) b' = x^{-1} \cdot a' b' g(1 \cdot x) = (a'b')^x \qquad (2.10.4)$$

for any $a' \in A' f(1)$ and any $x \in G$; it follows that $f(1)$ and $g(1)$ are $(B'^G)^\times$-conjugate; thus, we may assume that $f(1) = g(1) = 1$ and then 2.10.4 implies that G fixes b' (see the proof of Lemma 3.7 of [15] for more detail).

2.11. Let A be a G-algebra extension and denote by B the structural subalgebra. An *embedded algebra extension* $(C, \tilde{g})$ *associated with a pointed group* H_β *on* B is a pair formed by an H-algebra extension C and an embedding $\tilde{g}: C \to \mathrm{Res}_H^G(A)$ such that $g(1) \in \beta$; as in 2.13.1 of [19], it is clear that

2.11.1. for any pointed group H_β *on* B *there is an embedded algebra extension* $(C, \tilde{g})$ *associated with* H_β*; moreover, for any embedded algebra extension* $(C', \tilde{g}')$ *associated with* H_β *there is a unique exoisomorphism* $\tilde{h}: C' \cong C$ *such that* $\tilde{g}' = \tilde{g} \circ \tilde{h}$.

Indeed, if $j \in \beta$ it suffices to consider the $\mathcal{O}$-algebra $C = jAj$ endowed with the subalgebra jBj and the group homomorphism $H \to C^\times$ mapping $y \in H$ upon $y \cdot j$, and the embedding $\tilde{g}$ determined by the inclusion $C \subset A$. Moreover, if $(C', \tilde{g}')$ is another embedded algebra extension associated with H_β, we may assume that $g'(1) = j$; then g' induces an H-algebra extension isomorphism $C' \cong C$ and the uniqueness is clear since we may assume that $g'(c') = h(c')$ for any $c' \in C'$. Usually *we denote by* $(A_\beta, \tilde{f}_\beta)$ *an embedded algebra extension associated with* H_β chosen once for ever, *and by* B_β *the structural subalgebra.* Notice that the argument above, which proves the existence and uniqueness of $\tilde{h}$, shows that

2.11.2. if H_β *is a pointed group on* B *and* $\tilde{g}: C \to \mathrm{Res}_H^G(A)$ *an* H*-algebra extension exomorphism such that* $g(1) \in \beta$*, there is a unique exomorphism* $\tilde{h}: C \to A_\beta$ *such that* $\tilde{g} = \tilde{f}_\beta \circ \tilde{h}$

(see [18], 2.10.1 for the corresponding result on interior G-algebras). Similarly, it is not difficult to prove that (cf. 2.7)

2.11.3. for any pointed groups H_β *and* K_γ *on* B *such that* $K \subset H$*, we have* $K_\gamma \subset H_\beta$ *if and only if there is a* K*-algebra extension exomorphism* $\tilde{f}_\gamma^\beta: A_\gamma \to \mathrm{Res}_K^H(A_\beta)$ *such that* $\tilde{f}_\gamma = \mathrm{Res}_K^H(\tilde{f}_\beta) \circ \tilde{f}_\gamma^\beta$*, and then* $\tilde{f}_\gamma^\beta$ *is a uniquely determined embedding*

(see [19], 2.13.2 for the corresponding result on interior G-algebras).

2.12. Let us recall the definition of *A-fusions* (cf. [17]) – that we only need in Sect. 1. Here we assume that $S = \varnothing$ and $B = A$: that is, G is just a finite group and A an interior G-algebra. Let H_β and K_γ be pointed groups on A; an *A-fusion* $\tilde{\varphi}$ *from* K_γ *to* H_β is a group exomorphism $\tilde{\varphi}: K \to H$ such that φ is injective and there is an interior K-algebra embedding $\tilde{f}_\varphi: A_\gamma \to \mathrm{Res}_\varphi(A_\beta)$ fulfilling

$$\mathrm{Res}_1^K(\tilde{f}_\gamma) = \mathrm{Res}_1^H(\tilde{f}_\beta) \circ \mathrm{Res}_1^K(\tilde{f}_\varphi)\,. \qquad (2.12.1)$$

We denote by $F_A(K_\gamma, H_\beta)$ the set of A-fusions from K_γ to H_β and set $F_A(H_\beta) = F_A(H_\beta, H_\beta)$ which is a subgroup of $\tilde{\mathrm{A}}\mathrm{ut}(H)$ (cf. [17], 2.6 and 2.7). We call *local fusion*

category of A the category where objects are the local pointed groups on A and the morphisms are the corresponding A-fusions (cf. [17], Def. 2.15). Moreover, let P_γ be a local pointed group on A and denote by $N_{A_\gamma^\times}(P)$ the normalizer in $A_\gamma^\times$ of the structural image of P; the action of $N_{A_\gamma^\times}(P)$ on this image induces a group homomorphism from $F_A(P_\gamma)$ to $N_{A_\gamma^\times}(P)/P\cdot(A_\gamma^P)^\times$ (cf. [17], Cor. 2.13) and, considering $N_{A_\gamma^\times}(P)/P\cdot(1+J(A_\gamma^P))$ as a central $k^\times$-extension of $N_{A_\gamma^\times}(P)/P\cdot(A_\gamma^P)^\times$, we set (cf. [19], 6.7 and 6.14)

$$\hat{F}_A(P_\gamma) = \mathrm{Res}_{F_A(P_\gamma)}(N_{A_\gamma^\times}(P)/P\cdot(1+J(A_\gamma^P))) \tag{2.12.2}$$

which does not depend on the choice of $\tilde{f}_\gamma\colon A_\gamma \to \mathrm{Res}_P^G(A)$ (cf. [19], Prop. 6.8). It is not difficult to modify Proposition 6.12 of [19], replacing G by $A^\times$ and P by its structural image $\bar{P}$ in $A^\times$, to prove that:

2.12.3. If $\hat{F}_A(P_\gamma) \cong k^\times \times F_A(P_\gamma)$, there is a group homomorphism

$$\sigma\colon N_{A^\times}(\bar{P}_\gamma) \to A(P_\gamma)^\times$$

such that $s_\gamma(a^b) = \sigma(a^b) = s_\gamma(a)^{\sigma(b)}$ for any $a \in (A^P)^\times$ and any $b \in N_{A^\times}(\bar{P}_\gamma)$.

By a theorem of Dade (cf. [7], (12) or [16]) this situation occurs in the following case:

2.12.4. If $A = \mathrm{End}_{\mathcal{O}}(M)$, where M is an $\mathcal{O}G$-module, and P stabilizes an $\mathcal{O}$-basis of A containing the unity then $\mathscr{L}\mathscr{P}_A(P) = \{\gamma\}$ and $\hat{F}_A(P_\gamma) \cong k^\times \times F_A(P_\gamma)$. In particular, if Q_δ is a local pointed group on A and for any $j \in \delta$ we consider $j\cdot M$ as an $\mathcal{O}Q$-module, an injective group exomorphism $\tilde{\varphi}\colon Q \to P$ is an A-fusion from Q_δ to P_γ if and only if $j\cdot M$ is a direct summand of $\mathrm{Res}_\varphi(M)$.

Indeed, the existence and uniqueness of γ follow from Corollary 5.8 of [18] and, together with Proposition 2.18 of [17], they imply the last statement.

2.13. Let H be a subgroup of G and C an H-algebra extension, and denote by D the structural subalgebra of C; the *induced G-algebra extension* $\mathrm{Ind}_H^G(C)$ is formed by the tensor product $\mathcal{O}G \otimes_{\mathcal{O}H} C \otimes_{\mathcal{O}H} \mathcal{O}G$ endowed with the distributive product fulfilling

$$(x\otimes c\otimes y)(x'\otimes c'\otimes y') = \begin{cases} x\otimes c\cdot yx'\cdot c'\otimes y' & \text{if } yx'\in H \\ 0 \qquad \text{otherwise} \end{cases}$$

for any $x, y, x', y' \in G$ and any $c, c' \in C$, with the subalgebra $\sum_{y\in G} y\otimes D\otimes y^{-1}$ and with the group homomorphism mapping $x\in G$ on $\sum_y xy\otimes 1\otimes y^{-1}$ where y runs over a set of representatives for G/H in G. Moreover, we denote by

$$\tilde{d}_H^G(C)\colon C \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(C) \tag{2.13.1}$$

the canonical embedding determined by the H-algebra extension homomorphism mapping $c\in C$ on $1\otimes c\otimes 1$ (see [19], 2.14 for the corresponding construction for interior G-algebras). Conversely,

2.13.2. if A is a G-algebra extension and $\tilde{g}\colon C \to \mathrm{Res}_H^G(A)$ an H-algebra extension embedding such that $1 = \mathrm{Tr}_H^G(g(1))$, $g(1)g(1)^x = 0$ for any $x\in G-H$ and $g(1)$

centralizes the structural subalgebra B of A, then there is a unique exoisomorphism $\tilde{f}\colon \mathrm{Ind}_H^G(C) \cong A$ such that $\tilde{g} = \mathrm{Res}_H^G(\tilde{f}) \circ \tilde{d}_H^G(C)$.

Indeed, notice first that, if such an exoisomorphism exists, we may assume that $f(x \otimes c \otimes y) = x \cdot g(c) \cdot y$ for any $x, y \in G$ and any $c \in C$; conversely, it is clear that there is an $\mathcal{O}$-module homomorphism $f\colon \mathrm{Ind}_H^G(C) \to A$ mapping $x \otimes c \otimes y$ upon $x \cdot g(c) \cdot y$ and it is easily checked that f is an interior G-algebra isomorphism. Moreover, as $g(D) = Bg(1)$ (cf. 2.10) and $B = \sum_{x \in G} Bg(1)^x$, we get $B = f(\sum_{x \in G} x \otimes D \otimes x^{-1})$. Finally, if $d \in (D^H)^\times$ we have $\mathrm{Tr}_H^G(g(d)) \in (B^G)^\times$ and $(x \cdot g(c) \cdot y)^{\mathrm{Tr}_H^G(g(d))} = x \cdot g(c) \cdot y$ for any $x, y \in G$ and any $c \in C$, which proves that $\tilde{f}$ does not depend on the choice of g in $\tilde{g}$.

2.14. Let A be a G-algebra extension, α a point of G on the structural subalgebra B of A, and P_γ a defect pointed group of G_α (cf. 2.7). The following statement gives only a weak generalization of the corresponding result on interior G-algebras (cf. [15], Prop. 3.6), but it is strong enough to show that the interior G-algebra structure of the G-algebra extension A_α can "almost" be computed from the interior P-algebra structure of the P-algebra extension A_γ (the "almost" depending on the so called *multiplicity module* of γ as explained in Lemma 9.9 of [19] for interior G-algebras).

2.14.1. If H_β is a pointed group on B such that $H_\beta \subset G_\alpha$ and $\alpha \subset \mathrm{Tr}_H^G(B^H \cdot \beta \cdot B^H)$ then there is a unique interior G-algebra embedding $\tilde{g}_\alpha^\beta\colon A_\alpha \to \mathrm{Ind}_H^G(A_\beta)$ such that the following diagram of interior H-algebra embeddings commutes

$$\begin{array}{ccc} \mathrm{Res}_H^G(A_\alpha) & \xrightarrow{\mathrm{Res}_H^G(\tilde{g}_\alpha^\beta)} & \mathrm{Res}_H^G \mathrm{Ind}_H^G(A_\beta) \\ {\scriptstyle \tilde{f}_\beta^\alpha}\uparrow & \nearrow & {\scriptstyle \tilde{d}_H^G(A_\beta)} \\ A_\beta & & \end{array}$$

In particular, restriction through f_β^α induces an equivalence between the categories of A_α-and A_β-modules.

Indeed, if $i \in \alpha$ there are $j \in \beta$ and $b', b'' \in B^H$ such that $ij = j = ji$ (cf. 2.7) and $i = \mathrm{Tr}_H^G(b'jb'')$ (since $\mathcal{O}i + iJ(B^G)i = iB^Gi = \mathrm{Tr}_H^G(iB^HjB^Hi)$); now we may assume that $A_\alpha = iAi$ and $A_\beta = jAj$, and that $\tilde{f}_\alpha$, $\tilde{f}_\beta$ and $\tilde{f}_\beta^\alpha$ are determined by the inclusions $jAj \subset iAi \subset A$ (cf. 2.11); then it is easily checked that the $\mathcal{O}$-module homomorphism $g\colon A_\alpha \to \mathrm{Res}_H^G \mathrm{Ind}_H^G(A_\beta)$ mapping $a \in A_\alpha$ upon $\sum_{x,y} x \otimes jb'' \cdot x^{-1} \cdot a \cdot y \cdot b'j \otimes y^{-1}$, where x and y run over a set R of representatives for G/H in G, is an interior G-algebra homomorphism[5]. Moreover, setting $c' = \sum_{x \in R} 1 \otimes x \cdot b'j \otimes x^{-1}$ and $c'' = \sum_{x \in R} x \otimes jb'' \cdot x^{-1} \otimes 1$, it is clear that $c'c'' = 1 \otimes 1 \otimes 1$ and $g(a) = c''(1 \otimes a \otimes 1)c'$ for any $a \in A_\beta$; therefore, it follows from Lemma 3.7 of [15] that $\tilde{d}_H^G(A_\beta) = \mathrm{Res}_H^G(\tilde{g}) \circ \tilde{f}_\beta^\alpha$ where $\tilde{d}_H^G(A_\beta)$ and $\tilde{f}_\beta^\alpha$ denote the corresponding interior H-algebra embeddings (cf. 2.11.3 and 2.13.1). Finally, the fact that $\tilde{g}$ is a uniquely determined embedding (of interior G-algebras) follows from 2.10.3 and [19], 2.3.4.

[5] This homomorphism, which simplifies the proof of Proposition 3.6 of [15], has been pointed out by C. Picaronny

2.15. In this paper we are mainly concerned with the following situation: we consider a set S, a finite S-permutation group G, a normal subgroup H of G acting trivially on S, the G-algebra extension A formed by the $\mathcal{O}$-algebra $\mathcal{O}G$ endowed with the canonical map from G and with the subalgebra $B = \mathcal{O}H$, and a point α of G on B. For our purposes we have to translate some results in Chap. III of [12] to the local category of β_α (cf. 2.8): we try to be concise rather than complete. First of all we translate some terminology. A *G_α-biexomorphism* (or a *(G_α, S)-biexomorphism* to avoid confusion) (φ, ψ) is a pair of elements of $E_{G,S}(Q_\delta, P_\gamma)$ (cf. 2.8) where Q_δ is a local pointed group on B such that $Q_\delta \subset G_\alpha$ and P_γ a defect pointed group of G_α (cf. 2.7), and we say that $|P|/|Q|$ is the *length* $l(\varphi, \psi)$ *of* (φ, ψ); if $\varphi = (Q_\delta, x, P_\gamma)_S$ and $\psi = (Q_\delta, y, P_\gamma)_S$, where $x, y \in G$ and P_γ contains both $(Q_\delta)^x$ and $(Q_\delta)^y$ (cf. 2.8), we set $(\varphi, \psi) = (Q_\delta, x, y, P_\gamma)_S$. Moreover, if $\eta \in E_{G,S}(Q_\delta, P_\gamma)$ we write

$$(\varphi, \psi) = (\varphi, \eta) + (\eta, \psi) \tag{2.15.1}$$

and for any $\lambda \in E_{G,S}(Q'_{\delta'}, Q_\delta)$ and any $\mu \in E_{G,S}(P_\gamma, P'_{\gamma'})$, where $Q'_{\delta'}$ and $P'_{\gamma'}$ are local pointed groups on B, we set

$$\lambda(\varphi, \psi)\mu = (\mu \circ \varphi \circ \lambda, \mu \circ \psi \circ \lambda)\,; \tag{2.15.2}$$

notice that $E_{G,S}(Q'_{\delta'}, Q_\delta) \neq \varnothing$ implies $Q'_{\delta'} \subset G_\alpha$, and that $E_{G,S}(P_\gamma, P'_{\gamma'}) \neq \varnothing$ if and only if $P'_{\gamma'}$ is a defect pointed group of G_α. The law of composition 2.15.1. being associative (when defined), we say that (φ, ψ) is a *linear combination* of a set $\mathfrak{C}$ of G_α-biexomorphisms if there are nonempty ordered families $\{\lambda_i\}_{i \in I}$ and $\{\mu_i\}_{i \in I}$ of G-exomorphisms and $\{(\varphi_i, \psi_i)\}_{i \in I}$ of elements of $\mathfrak{C}$ such that we have

$$(\varphi, \psi) = \sum_{i \in I} \lambda_i(\varphi_i, \psi_i)\mu_i \tag{2.15.3}$$

in the obvious sense. Then we say that (φ, ψ) is *reducible* if this G_α-biexomorphism is a linear combination of the set of all G_α-biexomorphisms of length smaller than $l(\varphi, \psi)$, and *irreducible* otherwise (notice that this terminology differs slightly from the terminology introduced in Chap. III of [12])

2.16. A *generator set of the G_α-biexomorphisms* is a set $\mathfrak{C}$ of G_α-biexomorphisms such that any G_α-biexomorphism (φ, ψ) is a linear combination of $\mathfrak{C}$; then it is quite clear that (see Lemma 5 of [12], Ch. III).

2.16.1. if $\mathfrak{C}$ is a generator set of the G_α-biexomorphisms then the set of irreducible elements of $\mathfrak{C}$ is also a generator set.

To find minimal generator sets of the G_α-biexomorphisms we need the following fact which generalizes Lemmas 3.9 and 3.10 of [18] and can be proved essentially by the same arguments. Notice that for any p-subgroup Q of G we have $B(Q) \cong \mathcal{k}C_H(Q)$ (cf. [19], 2.8.4) and therefore any simple $\mathcal{k}C_H(Q)$-module is associated with a local point of Q on B.

2.16.2. If Q_δ is a local pointed group on B and K a subgroup of G containing $Q \cdot C_H(Q)$, there is a unique point ν of K on B such that $Q_\delta \subset K_\nu$. Then Q_δ is a defect pointed group of K_ν if and only if $N_K(Q_\delta)/Q \cdot C_H(Q)$ is a p'-group and the simple $\mathcal{k}(C_H(Q)/(H \cap Z(Q)))$-module associated with δ is projective, and in that case $\mathrm{Br}_Q^B(\delta)$

is the unique point of $B(Q)f$ where f is the primitive idempotent of $Z(B(Q))$ such that $f\,\mathrm{Br}_Q^B(\delta) = \mathrm{Br}_Q^B(\delta)$. In particular, if L is a normal subgroup of K containing $C_H(Q)$ and K/L is a p-group, we have $K = R \cdot L$ where R_ε is a defect pointed group of K_ν.

Indeed, setting $\bar{C} = Q \cdot C_H(Q)/Q$, we have $B(Q_\delta)^{\bar{C}} \cong \mathcal{k}$ and therefore $s_\delta(B^K) \cong \mathcal{k}$ which proves the uniqueness of ν. On the other hand, setting $\bar{N} = N_K(Q_\delta)/Q$, we have $B(Q_\delta)_1^{\bar{N}} = |\bar{N}/\bar{C}|B(Q_\delta)_1^{\bar{C}}$ and if $B(Q_\delta)_1^{\bar{C}} \cong \mathcal{k}$ then the image $\bar{f}$ of f in $Z(\mathcal{k}\bar{C})$ determines a block of defect zero of $\bar{C}$ which implies $|\mathscr{P}(B(Q)f)| = 1$. Moreover, setting $M = R \cdot L$, we get easily $N_K(M) = L \cdot (N_G(R_\varepsilon) \cap N_K(M))$ (since we have $R_\varepsilon \subset M_\lambda \lhd N_K(M)_\mu \subset K_\nu$), and therefore $N_K(M)/M$ is both a p-group and a p'-group; so, as M is subnormal in K, we get $M = K$.

2.17. Let P_γ be a defect pointed group of G_α; as G acts transitively on the set of them (cf. [19], 2.10.2), the set of all the G_α-biexomorphisms $(Q_\delta, 1, x, P_\gamma)_S$ is a generator set. First of all, we have the following reduction:

2.17.1. The set of all the G_α-biexomorphisms $(Q_\delta, 1, n, P_\gamma)_S$ such that $n \in N_G(Q_\delta)$ and P_γ contains a defect pointed group of $N_G(Q_\delta)_\nu$, where ν is the point fulfilling $Q_\delta \subset N_G(Q_\delta)_\nu$, is a generator set.

By 2.16.1 it suffices to prove that any *G_α-biexomorphism* $(Q_\delta, 1, x, P_\gamma)_S$ is a linear combination of the union of the set of reducible G_α-biexomorphisms and the set above; moreover, we may assume that $Q \neq P$ and then there are local pointed groups $Q'_{\delta'}$ and $Q''_{\delta''}$ on B such that (cf. [15], cor. 1.5)

$$Q_\delta \subsetneqq Q'_{\delta'} \subset P_\gamma \quad \text{and} \quad (Q_\delta)^x \subsetneqq (Q''_{\delta''})^x \subset P_\gamma\,.$$

Let ν be the unique point of $N_G(Q_\delta)$ on B such that $N_G(Q_\delta)_\nu$ contains Q_δ, $Q'_{\delta'}$ and $Q''_{\delta''}$ (cf. 2.16.3), R_ε a defect pointed group of $N_G(Q_\delta)_\nu$ containing $Q'_{\delta'}$ (cf. 2.7), y an element of G such that $(R_\varepsilon)^y \subset P_\gamma$ (cf. [19], 2.10.2) and n an element of $N_G(Q_\delta)$ such that $(Q''_{\delta''})^n \subset R_\varepsilon$ (cf. [19], 2.10.2); now it is clear that

$$(Q_\delta, 1, x, P_\gamma)_S =$$
$$(Q_\delta, 1, y, P_\gamma)_S + (Q_\delta, y, (Q_\delta)^y)_S((Q_\delta)^y, 1, n^y, P_\gamma)_S + (Q_\delta, ny, x, P_\gamma)_S$$

and that the G_α-biexomorphisms $(Q_\delta, 1, y, P_\gamma)_S$ and $(Q_\delta, ny, x, P_\gamma)_S$ are reducible (since $Q'_{\delta'} \neq Q_\delta \neq Q''_{\delta''}$).

2.17.2. If $(Q_\delta, 1, x, P_\gamma)_S$ is an irreducible G_α-biexomorphism then Q_δ is a defect pointed group of $Q \cdot C_G(Q_\delta, S)_\mu$, where μ is the unique point such that $Q_\delta \subset Q \cdot C_G(Q_\delta, S)_\mu$.

Indeed, by 2.17.1 we may assume that $x \in N_G(Q_\delta)$ and that P_γ contains a defect pointed group R_ε of $Q \cdot C_G(Q_\delta, S)_\mu$ (notice that $Q_\delta \subset R_\varepsilon$); then we have $x = mz$ where $m \in N_G(R_\varepsilon)$ and $z \in C_G(Q_\delta, S)$ (cf. [19], 2.10.2), and therefore $(Q_\delta, 1, x, P_\gamma)_S = (Q_\delta, 1, R_\varepsilon)_S(R_\varepsilon, 1, m, P_\gamma)_S$ which forces $Q_\delta = R_\varepsilon$.

2.18. Let Q_δ be a local pointed group on B and μ the unique point of $Q \cdot C_G(Q_\delta, S)$ on B such that $Q_\delta \subset Q \cdot C_G(Q_\delta, S)_\mu$ (cf. 2.16.2). Now *we assume that Q_δ is properly contained in P_γ and a defect pointed group of $Q \cdot C_G(Q_\delta, S)_\mu$*. First of all, notice that

2.18.1. we have $C_P(Q_\delta, S) \subset Q$.

Indeed, as H is a G-stable $\mathcal{O}$-basis of B, there is a local point ε of $Q \cdot C_P(Q_\delta, S)$ on B such that $Q \cdot C_P(Q_\delta, S)_\varepsilon \subset P_\gamma$, and then a local point δ' of Q on B such that $Q_{\delta'} \subset Q \cdot C_P(Q_\delta, S)_\varepsilon$ (cf. [1], Prop. 1.5); but, by 2.16.2 above and Theorem 1.8 of [1], we have $\delta' = \delta$ and therefore $Q \cdot C_P(Q_\delta, S)_\varepsilon \subset Q \cdot C_G(Q_\delta, S)_\mu$ (cf. 2.16.2); consequently, $Q = Q \cdot C_P(Q_\delta, S)$ (cf. 2.17.2). Recall that $E_{G,S}(Q_\delta)$ is isomorphic to $N_G(Q_\delta)/Q \cdot C_G(Q_\delta, S)$ (cf. 2.8) and denote by $\bar{X}$ the image of $X \subset N_G(Q_\delta)$ in $E_{G,S}(Q_\delta)$.

2.18.2. A G_α*-biexomorphism* $(Q_\delta, 1, x, P_\gamma)_S$ *is reducible if and only if there are p-subgroups* $\bar{R}_0, \ldots, \bar{R}_n$ *of* $E_{G,S}(Q_\delta)$ *such that* $\overline{N_P(Q_\delta)} \subset \bar{R}_0$, $\overline{N_{P_{x^{-1}}}(Q_\delta)} \subset \bar{R}_n$ *and* $\bar{R}_{i-1} \cap \bar{R}_i \neq \{1\}$ *if* $1 \leqq i \leqq n$.

Indeed, if $(Q_\delta, 1, x, P_\gamma)_S$ is reducible, there are $\varphi_0, \ldots, \varphi_n \in E_{G,S}(Q_\delta)$ such that $\varphi_0 = (Q_\delta, 1, P_\gamma)_S$, $\varphi_n = (Q_\delta, x, P_\gamma)_S$ and if $1 \leqq i \leqq n$ we have $(\varphi_{i-1}, \varphi_i) = \lambda_i(\psi_{i-1}, \psi_i')$ for some G_α-biexomorphism (ψ_{i-1}, ψ_i') such that $l(\psi_{i-1}, \psi_i') < |P|/|Q|$ (cf. 2.15); then, by Corollary 1.5 of [15] and 2.18.1 above, it suffices to set $\bar{R}_i = \overline{N_{P_{x_i^{-1}}}(Q_\delta)}$ if $\varphi_i = (Q_\delta, x_i, P_\gamma)_S$, since $\bar{R}_i$ does not depend on the choice of x_i (cf. 2.8, 2.16.2 and [1], Th. 1.8). Conversely, assume that there are $\bar{R}_0, \ldots, \bar{R}_n$ as in 2.18.2; we may assume that $\bar{R}_0 = \overline{N_P(Q_\delta)}$ and $\bar{R}_n = \overline{N_{P_{x^{-1}}}(Q_\delta)}$; if $0 \leqq i \leqq n$ denote by K_i the inverse image of $\bar{R}_i$ in $N_G(Q_\delta)$, by v_i the point of K_i on B such that $Q_\delta \subset (K_i)_{v_i}$, and by x_i an element of G such that $(P_\gamma)^{x_i^{-1}}$ contains a defect pointed group $(R_i)_{\varepsilon_i}$ of $(K_i)_{v_i}$; notice that $K_i = R_i \cdot C_G(Q_\delta, S)$ (2.16.2). Now, as we may assume that $x_0 = 1$ and $x_n = x$ (cf. 2.16.2 and [1], Prop. 1.5 and Th. 1.8), it suffices to prove that $(Q_\delta, x_{i-1}, x_i, P_\gamma)_S$ is reducible for any $1 \leqq i \leqq n$; fix i and let K be the inverse image of $\bar{R}_{i-1} \cap \bar{R}_i$ in $N_G(Q_\delta)$, v the point of K on B such that $Q_\delta \subset K_v$ and R_ε a defect pointed group of K_v; as $K_v \subset (K_{i-1})_{v_{i-1}}$ and $K_v \subset (K_i)_{v_i}$ (cf. 2.16.2), there are $z_{i-1}, z_i \in C_G(Q_\delta, S)$ such that (cf. [19], 2.10.2)

$$(R_\varepsilon)^{z_{i-1}} \subset (R_{i-1})_{\varepsilon_{i-1}} \qquad \text{and} \qquad (R_\varepsilon)^{z_i} \subset (R_i)_{\varepsilon_i}$$

and therefore we have

$$(Q_\delta, x_{i-1}, x_i, P_\gamma)_S = (Q_\delta, 1, R_\varepsilon)_S (R_\varepsilon, z_{i-1}x_{i-1}, z_i x_i, P_\gamma)_S ;$$

but $Q_\delta \neq R_\varepsilon$ since $\bar{R}_{i-1} \cap \bar{R}_i \neq \{1\}$ (cf. 2.16.2), which completes the proof.

Then, denoting by $\bar{M}$ the minimal subgroup of $E_{G,S}(Q_\delta)$ which contains $\overline{N_P(Q_\delta)}$ and fulfils the following condition (cf. [12], Ch. II, Prop. 1).

2.18.3. for any $\bar{n} \in E_{G,S}(Q_\delta) - \bar{M}$ *the intersection* $\bar{M} \cap \bar{M}^{\bar{n}}$ *is a p'-group.*

It follows from Proposition 2 and Theorem 1 of Chap. III of [12], and from 2.18.2 above that (cf. 2.8)

2.18.4. a G_α*-biexomorphism* $(Q_\delta, 1, x, P_\gamma)_S$ *such that* $x \in N_G(Q_\delta)$ *is irreducible if and only if* $\bar{x} \in E_{G,S}(Q_\delta) - \bar{M}$, *and then we have* $Q = P \cap P^x$ *and* $C_G^p(Q_\delta, S) = Q \cdot C_G(Q_\delta, S)$.

2.19. Finally, *assume that* $H = \{1\}$ *and that S is endowed with a p'-divisible abelian group structure compatible with the action of G*; in that case all the points considered

above coincide with $\{1\}$ and we omit to mention them; so, P is just a Sylow p-subgroup of G. For any subgroup K of G we denote by $\mathbb{H}^n(K, S)$ the corresponding n-th cohomology group (cf. [4], Ch. XII, §2), and it is quite clear that any G-exomorphism $\varphi \in E_{G,S}(K, L)$ induces a group homomorphism $\mathbb{H}^n(\varphi)$ from $\mathbb{H}^n(L, S)$ to $\mathbb{H}^n(K, S)$ (cf. [4], Ch. XII, §8). In particular, the inclusion $P \subset G$ induces an injective group homomorphism $\mathbb{H}^n(G, S) \to \mathbb{H}^n(P, S)$ and its image is the intersection of the kernels

$$\mathbb{K}^n((\varphi, \psi)) = \mathrm{Ker}(\mathbb{H}^n(\varphi) - \mathbb{H}^n(\psi)) \tag{2.19.1}$$

where (φ, ψ) runs over the set of all the G-biexomorphisms from any subgroup Q of P to P (cf. [4], Ch. XII, Th. 10.1). Now, the argument of Lemma 9 of [12] proves that

2.19.2. If X is a subset of G containing $N_G(P)$ such that the family $\{(P \cap P^{x^{-1}}, 1, x, P)_S\}_{x \in X}$ is a generator set of the G-biexomorphisms, the inclusion $P \subset G$ induces an isomorphism

$$\mathbb{H}^n(G, S) \cong \bigcap_{x \in X} \mathbb{K}^n((P \cap P^{x^{-1}}, 1, x, P)_S) .$$

3. Semicovering exomorphisms

3.1. An important tool to prove Theorem 1.12 when G/H is a p-group is a class of G-algebra exomorphisms that we name *semicovering exomorphisms*. This class allows us to lift pointed p-groups (see Proposition 3.15 below). Our approach tries to make evident the analogy with the *covering exomorphisms* introduced in Sect. 4 of [18], the main difference coming from the fact that Theorem 3.13 below fails beyond the set of p-subgroups.

3.2. First of all, it is handy to consider the $\mathcal{O}$-algebra case (i.e. the case where $G = \{1\}$). Let A and A' be $\mathcal{O}$-algebras, $\mathfrak{a}$ and $\mathfrak{a}'$ respective ideals of A and A' (cf. 2.2), and $\tilde{f}: A \to A'$ an $\mathcal{O}$-algebra exomorphism (cf. 2.3) such that $f(\mathfrak{a}) \subset \mathfrak{a}'$. We say that $\tilde{f}$ is a *semicovering exomorphism from $\mathfrak{a}$ to $\mathfrak{a}'$* if $f(J(\mathfrak{a})) \subset J(\mathfrak{a}')$ and $f(\mathfrak{a})$ contains a maximal commutative $\mathcal{O}$-semisimple subalgebra of $\mathfrak{a}'$ (cf. 2.2 and [14], §1) or, equivalently,

$$\sum_{\alpha \in \mathscr{P}(\mathfrak{a}) - \mathscr{P}(\mathrm{Ker}(f))} m_\alpha = \sum_{\alpha' \in \mathscr{P}(\mathfrak{a}')} m_{\alpha'} . \tag{3.2.1}$$

Then we say that $\tilde{f}$ is *strict on* $\mathfrak{a}$ if moreover $\mathfrak{a} \cap \mathrm{Ker}(f) \subset J(\mathfrak{a})$ or, equivalently, $\mathscr{P}(\mathfrak{a}) \cap \mathscr{P}(\mathrm{Ker}(f)) = \varnothing$. Notice that, if $\tilde{f}$ is a semicovering exomorphism from $\mathfrak{a}$ to $\mathfrak{a}'$, then $f(\mathfrak{a})$ contains all the idempotents of $\mathfrak{a}' \cap Z(A')$. Moreover, it is clear that $\tilde{f}$ is a semicovering exomorphism from $\mathfrak{a}$ to $\mathfrak{a}'$ if and only if $\mathcal{k} \otimes \tilde{f}: \mathcal{k} \otimes_{\mathcal{O}} A \to \mathcal{k} \otimes_{\mathcal{O}} A'$ is a semicovering exomorphism from $\bar{\mathfrak{a}}$ to $\bar{\mathfrak{a}}'$ (cf. 2.1), and then $\tilde{f}$ is strict on $\mathfrak{a}$ if and only if $\mathcal{k} \otimes \tilde{f}$ is strict on $\bar{\mathfrak{a}}$. When $\mathfrak{a} = A$ and $\mathfrak{a}' = A'$ we say just that $\tilde{f}$ is a (*strict*) *semicovering exomorphism* and then $f(1) = 1$.

Proposition 3.3. *The following conditions on $\tilde{f}$ are equivalent:*

3.3.1. The exomorphism $\tilde{f}: A \to A'$ is a semicovering from $\mathfrak{a}$ to $\mathfrak{a}'$.

3.3.2. We have $f(J(\mathfrak{a})) \subset J(\mathfrak{a}')$ *and for any* $\alpha' \in \mathscr{P}(\mathfrak{a}')$ *there is* $\mathscr{S}(\alpha') \subset \mathscr{P}(\mathfrak{a})$ *such that* $m_{\alpha'} = \sum_{\alpha \in \mathscr{S}(\alpha')} m_\alpha$ *and* $f(\alpha) \subset \alpha'$ *for any* $\alpha \in \mathscr{S}(\alpha')$.

3.3.3. We have $f(J(\mathfrak{a})) \subset J(\mathfrak{a}')$ *and there is a map* $\tilde{f}^*: \mathbb{S}(\mathscr{P}(\mathfrak{a}')) \to \mathbb{S}(\mathscr{P}(\mathfrak{a}))$, *preserving union and intersection, such that, for any* $\mathscr{S}' \subset \mathscr{P}(\mathfrak{a}')$, *we have* $\sum_{\alpha' \in \mathscr{S}'} m_{\alpha'} = \sum_{\alpha \in \tilde{f}^*(\mathscr{S}')} m_\alpha$ *and* $\tilde{f}$ *induces an injective unitary exomorphism*

$$\prod_{\alpha \in \tilde{f}^*(\mathscr{S}')} A(\alpha) \to \prod_{\alpha' \in \mathscr{S}'} A'(\alpha') .$$

In that case, we have $\mathscr{P}(f^{-1}(\mathfrak{a}')) - \tilde{f}^*(\mathscr{P}(\mathfrak{a}')) = \mathscr{P}(\mathrm{Ker}(f))$ *and, in particular,* $\tilde{f}$ *is strict on* $\mathfrak{a}$ *if and only if* $\tilde{f}^*(\mathscr{P}(\mathfrak{a}')) = \mathscr{P}(\mathfrak{a})$.

Remark 3.4. If $\tilde{f}$ is a semicovering exomorphism from $\mathfrak{a}$ to $\mathfrak{a}'$, it is clear that $\tilde{f}^*(\mathscr{S}')$ is the disjoint union $\bigcup_{\alpha' \in \mathscr{S}'} \mathscr{S}(\alpha')$ for any $\mathscr{S}' \subset \mathscr{P}(A')$, and that $\mathscr{S}(\alpha')$ is the set of $\alpha \in \mathscr{P}(A)$ such that $m(f)^\alpha_{\alpha'} \neq 0$ (cf. 2.3) for any $\alpha' \in \mathscr{P}(\mathfrak{a}')$ (since $m_{\alpha'} \geqq \sum_{\alpha \in \mathscr{P}(A)} m(f)^\alpha_{\alpha'} m_\alpha$).

Proof. If $\tilde{f}$ is a semicovering exomorphism from $\mathfrak{a}$ to $\mathfrak{a}'$ and T is a maximal commutative $\mathcal{O}$-semisimple subalgebra of $\mathfrak{a}$, then $f(T)$ is a maximal commutative $\mathcal{O}$-semisimple subalgebra of $\mathfrak{a}'$ (cf. [14], §1) and, in particular, for any primitive idempotent i of T such that $f(i) \neq 0$, $f(i)$ is a primitive idempotent of $\mathfrak{a}'$; thus, for any $\alpha' \in \mathscr{P}(\mathfrak{a}')$, we have

$$m_{\alpha'} = |\alpha' \cap f(T)| = \sum_\alpha |\alpha \cap T| = \sum_\alpha m_\alpha$$

where α runs over the set of points of $\mathfrak{a}$ such that $f(\alpha) \subset \alpha'$.

Assume now that 3.3.2 holds. To prove 3.3.3 it suffices to consider $\tilde{f}^*: \mathbb{S}(\mathscr{P}(\mathfrak{a}')) \to \mathbb{S}(\mathscr{P}(\mathfrak{a}))$ mapping $\mathscr{S}' \subset \mathscr{P}(\mathfrak{a}')$ upon the disjoint union $\bigcup_{\alpha' \in \mathscr{S}'} \mathscr{S}(\alpha')$. Indeed, it is clear that such a map preserves union and intersection; moreover, we have

$$\sum_{\alpha' \in \mathscr{S}'} m_{\alpha'} = \sum_{\alpha' \in \mathscr{S}'} \sum_{\alpha \in \mathscr{S}(\alpha')} m_\alpha = \sum_{\alpha \in \tilde{f}^*(\mathscr{S}')} m_\alpha \tag{3.3.5}$$

and, in particular, $\sum_{\alpha' \in \mathscr{P}(\mathfrak{a}')} m_{\alpha'} = \sum_{\alpha \in \tilde{f}^*(\mathscr{P}(\mathfrak{a}'))} m_\alpha$ which proves that $\mathscr{P}(f^{-1}(\mathfrak{a}')) - \tilde{f}^*(\mathscr{P}(\mathfrak{a}')) = \mathscr{P}(\mathrm{Ker}(f))$ (since for any $\alpha \in \tilde{f}^*(\mathscr{P}(\mathfrak{a}'))$ there is $\alpha' \in \mathscr{P}(\mathfrak{a}')$ such that $f(\alpha) \subset \alpha'$, whereas if $\alpha \in \mathscr{P}(f^{-1}(\mathfrak{a}')) - \tilde{f}^*(\mathscr{P}(\mathfrak{a}'))$, it follows from 3.3.5 that $m(f)^\alpha_{\alpha'} = 0$ for any $\alpha' \in \mathscr{P}(A')$); finally, as $f(J(\mathfrak{a})) \subset J(\mathfrak{a}')$, $\tilde{f}$ induces a unitary exomorphism

$$\prod_{\alpha \in \mathscr{P}(\mathfrak{a})} A(\alpha) \to \prod_{\alpha' \in \mathscr{P}(\mathfrak{a}')} A'(\alpha') \to \prod_{\alpha' \in \mathscr{S}'} A'(\alpha')$$

and the kernel is clearly equal to $\prod_{\alpha \in \mathscr{P}(\mathfrak{a}) - \tilde{f}^*(\mathscr{S}')} A(\alpha)$.

If 3.3.3 holds and T and T' are respectively maximal commutative $\mathcal{O}$-semisimple subalgebras of $\mathfrak{a}$ and $\mathfrak{a}'$ such that $f(T) \subset T'$, we have

$$\mathrm{rank}_{\mathcal{O}}(T') = \sum_{\alpha' \in \mathscr{P}(\mathfrak{a}')} m_{\alpha'} = \sum_{\alpha \in \tilde{f}^*(\mathscr{P}(\mathfrak{a}'))} m_\alpha \leqq \mathrm{rank}_{\mathcal{O}}(f(T)) ,$$

and therefore $f(T) = T'$.

Proposition 3.5. *Let* $\mathfrak{b}$ *and* $\mathfrak{b}'$ *be respectively ideals of* A *and* A' *such that* $f(\mathfrak{b}) \subset \mathfrak{b}'$. *If* $\tilde{f}$ *is a semicovering exomorphism from* $\mathfrak{a}$ *to* $\mathfrak{a}'$ *and from* $\mathfrak{b}$ *to* $\mathfrak{b}'$ *then* $\tilde{f}$ *is*

also a semicovering exomorphism from $\mathfrak{a}+\mathfrak{b}$ *to* $\mathfrak{a}'+\mathfrak{b}'$ *and the associated map* $\tilde{f}^*\colon \mathbb{S}(\mathscr{P}(\mathfrak{a}'+\mathfrak{b}'))\to\mathbb{S}(\mathscr{P}(\mathfrak{a}+\mathfrak{b}))$ *extends the associated maps*

$$\mathbb{S}(\mathscr{P}(\mathfrak{a}'))\to\mathbb{S}(\mathscr{P}(\mathfrak{a}))\qquad\text{and}\qquad\mathbb{S}(\mathscr{P}(\mathfrak{b}'))\to\mathbb{S}(\mathscr{P}(\mathfrak{b}))\,.\tag{3.5.1}$$

If moreover $\tilde{f}$ *is strict on* $\mathfrak{a}$ *and* $\mathfrak{b}$ *then* $\tilde{f}$ *is also strict on* $\mathfrak{a}+\mathfrak{b}$.

Proof. Since $J(\mathfrak{a}+\mathfrak{b})=J(\mathfrak{a})+J(\mathfrak{b})$, if $\tilde{f}$ is a semicovering exomorphism from $\mathfrak{a}$ to $\mathfrak{a}'$ and from $\mathfrak{b}$ to $\mathfrak{b}'$, we have $f(J(\mathfrak{a}+\mathfrak{b}))\subset J(\mathfrak{a}'+\mathfrak{b}')$. On the other hand, as $\mathscr{P}(\mathfrak{a}'+\mathfrak{b}')=\mathscr{P}(\mathfrak{a}')\cup\mathscr{P}(\mathfrak{b}')$, it follows from 3.3.2 that for any $\alpha'\in\mathscr{P}(\mathfrak{a}'+\mathfrak{b}')$ there is $\mathscr{S}(\alpha')\subset\mathscr{P}(\mathfrak{a}+\mathfrak{b})$ such that $m_{\alpha'}=\sum_{\alpha\in\mathscr{S}(\alpha')}m_\alpha$ and $f(\alpha)\subset\alpha'$ for any $\alpha\in\mathscr{S}(\alpha')$. Thus, by Proposition 3.3, $\tilde{f}$ is a semicovering exomorphism from $\mathfrak{a}+\mathfrak{b}$ to $\mathfrak{a}'+\mathfrak{b}'$; Then, by Remark 3.4, $\tilde{f}^*\colon\mathbb{S}(\mathscr{P}(\mathfrak{a}'+\mathfrak{b}'))\to\mathbb{S}(\mathscr{P}(\mathfrak{a}+\mathfrak{b}))$ maps $\{\alpha'\}$ upon $\mathscr{S}(\alpha')$ and therefore extends the associated maps 3.5.1. Finally, the last statement follows from the last statement of Proposition 3.3.

Proposition 3.6. *Let* A'' *be an* $\mathcal{O}$*-algebra,* $\mathfrak{a}''$ *an ideal of* A'' *and* $\tilde{g}\colon A'\to A''$ *an exomorphism such that* $g(\mathfrak{a}')\subset\mathfrak{a}''$.

3.6.1. If $\tilde{f}$ *and* $\tilde{g}$ *are respectively semicovering exomorphisms from* $\mathfrak{a}$ *to* $\mathfrak{a}'$ *and from* $\mathfrak{a}'$ *to* $\mathfrak{a}''$ *then* $\tilde{g}\circ\tilde{f}$ *is a semicovering exomorphism from* $\mathfrak{a}$ *to* $\mathfrak{a}''$ *and we have* $(\tilde{g}\circ\tilde{f})^*=\tilde{f}^*\circ\tilde{g}^*$. *In that case, if two semicovering exomorphisms among* $\tilde{f}$, $\tilde{g}$ *and* $\tilde{g}\circ\tilde{f}$ *are respectively strict on* $\mathfrak{a}$, $\mathfrak{a}'$ *and* $\mathfrak{a}$, *so is the third.*

3.6.2. If $\tilde{g}\circ\tilde{f}$ *is a semicovering exomorphism from* $\mathfrak{a}$ *to* $\mathfrak{a}''$ *and* $g(J(\mathfrak{a}'))\subset J(\mathfrak{a}'')$ *then* $\tilde{g}$ *is a semicovering from* $\mathfrak{a}'$ *to* $\mathfrak{a}''$. *If moreover* $\tilde{g}$ *is strict on* $\mathfrak{a}'$ *then* $\tilde{f}$ *is a semicovering from* $\mathfrak{a}$ *to* $\mathfrak{a}'$.

Proof. If $\tilde{f}$ and $\tilde{g}$ are respectively semicovering exomorphisms from $\mathfrak{a}$ to $\mathfrak{a}'$ and from $\mathfrak{a}'$ to $\mathfrak{a}''$, we have $g(f(J(\mathfrak{a})))\subset g(J(\mathfrak{a}'))\subset J(\mathfrak{a}'')$ and, for any maximal commutative $\mathcal{O}$-semisimple subalgebra T of $\mathfrak{a}$, $f(T)$ is a maximal commutative $\mathcal{O}$-semisimple subalgebra of $\mathfrak{a}'$, whence $g(f(T))$ is a maximal commutative $\mathcal{O}$-semisimple subalgebra of $\mathfrak{a}''$. Conversely, if $g(f(T))$ and T' are respectively maximal commutative $\mathcal{O}$-semisimple subalgebras of $\mathfrak{a}''$ and $\mathfrak{a}'$ in such a way that $f(T)\subset T'$, it is clear that $g(T')=g(f(T))$ which proves the first statement in 3.6.2. If moreover $\mathfrak{a}'\cap\mathrm{Ker}(g)\subset J(\mathfrak{a}')$, the intersection $\mathfrak{a}'\cap g^{-1}(J(\mathfrak{a}''))$ has no nonzero idempotents and therefore we get $f(J(\mathfrak{a}))\subset\mathfrak{a}'\cap g^{-1}(J(\mathfrak{a}''))\subset J(\mathfrak{a}')$; on the other hand, as $T'\cap\mathrm{Ker}(g)=\{0\}$, the equality $g(T')=g(f(T))$ and the inclusion $f(T)\subset T'$ imply that $f(T)=T'$.

Corollary 3.7. *Let* $\bar{A}$ *and* $\bar{A}'$ *be respectively quotient algebras of* A *and* A' *such that* f *induces a homomorphism* $\bar{f}\colon\bar{A}\to\bar{A}'$, *and denote respectively by* $\bar{\mathfrak{a}}$ *and* $\bar{\mathfrak{a}}'$ *the images of* $\mathfrak{a}$ *in* $\bar{A}$ *and* $\mathfrak{a}'$ *in* $\bar{A}'$. *If* $\tilde{f}$ *is a semicovering exomorphism from* $\mathfrak{a}$ *to* $\mathfrak{a}'$ *then* $\bar{\tilde{f}}$ *is a semicovering exomorphism from* $\bar{\mathfrak{a}}$ *to* $\bar{\mathfrak{a}}'$.

Proof. By 3.6.1, the exomorphism $A\to\bar{A}'$ induced by $\tilde{f}$ is a semicovering from $\mathfrak{a}$ to $\bar{\mathfrak{a}}'$ and then, by 3.6.2, the exomorphism $\bar{\tilde{f}}$ is a semicovering from $\bar{\mathfrak{a}}$ to $\bar{\mathfrak{a}}'$ (since $J(\bar{\mathfrak{a}})$ is the image of $J(\mathfrak{a})$ in $\bar{A}$).

Proposition 3.8. *Let* B *and* B' *be* $\mathcal{O}$*-algebras,* $\tilde{e}\colon B\to A$ *and* $\tilde{e}'\colon B'\to A'$ $\mathcal{O}$*-algebra embeddings and* $\tilde{g}\colon B\to B'$ *a unitary exomorphism such that* $\tilde{e}'\circ\tilde{g}=\tilde{f}\circ\tilde{e}$. *Set*

$\mathfrak{b} = e^{-1}(\mathfrak{a})$ *and* $\mathfrak{b}' = e'^{-1}(\mathfrak{a}')$. *If* $\tilde{f}$ *is a semicovering exomorphism from* $\mathfrak{a}$ *to* $\mathfrak{a}'$ *then* $\tilde{g}$ *is a semicovering exomorphism from* $\mathfrak{b}$ *to* $\mathfrak{b}'$. *If moreover* $\tilde{f}$ *is strict on* $\mathfrak{a}$ *then* $\tilde{g}$ *is strict on* $\mathfrak{b}$.

Proof. Assume that $e' \circ g = f \circ e$ and choose maximal commutative $\mathcal{O}$-semisimple subalgebras T of $\mathfrak{a}$ and T' of $\mathfrak{a}'$ such that T centralizes $e(1)$, T' centralizes $e'(1)$ and $f(T) \subset T'$ (which is always possible!); then $e^{-1}(T)$ and $e'^{-1}(T')$ are respectively maximal commutative $\mathcal{O}$-semisimple subalgebras of $\mathfrak{b}$ and $\mathfrak{b}'$, and we have $g(e^{-1}(T)) \subset e'^{-1}(T')$; moreover, it is clear that $f(T) = T'$ implies $g(e^{-1}(T)) = e'^{-1}(T')$. On the other hand, we have $e^{-1}(J(\mathfrak{a})) = J(\mathfrak{b})$ and $e'^{-1}(J(\mathfrak{a}')) = J(\mathfrak{b}')$.

Example 3.9. Let G and G' be finite groups and $\varphi: G \to G'$ a group homomorphism such that $\mathrm{Im}(\varphi)$ is normal in G' and $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are both p-groups, and denote by $f: \mathcal{O}G \to \mathcal{O}G'$ the $\mathcal{O}$-algebra homomorphism induced by φ. Then $\tilde{f}$ is a *strict semicovering exomorphism.* Indeed, since $\mathrm{Ker}(\varphi)$ is a p-group, we have $\mathrm{Ker}(f) \subset J(\mathcal{O}G)$ and, as $\mathrm{Im}(\varphi)$ is normal in G', we get $f(J(\mathcal{O}G)) \subset J(\mathcal{O}G')$; finally, if P' is a Sylow p-subgroup of G', $\mathrm{Im}(f)$ is P'-stable and we have $\mathcal{O}G' = \sum_{u' \in P'} \mathrm{Im}(f)u'$ which implies that a maximal commutative $\mathcal{O}$-semisimple subalgebra of $\mathrm{Im}(f)$ is still maximal in $\mathcal{O}G'$ (cf. [14], Th. 2).

3.10. Let G be a finite group: we are ready to discuss the G-algebra case. Let B and B' be G-algebras and $\tilde{g}: B \to B'$ a G-algebra exomorphism (cf. 2.9 and 2.10). We say that $\tilde{g}$ is a *semicovering exomorphism* if, for any p-subgroup P of G, the induced $\mathcal{O}$-algebra exomorphism $\tilde{g}^P: B^P \to B'^P$ is a semicovering (cf. 3.2), and then we denote by

$$(\tilde{g}^P)^*: \mathbb{S}(\mathscr{P}_{B'}(P)) \to \mathbb{S}(\mathscr{P}_B(P)) \tag{3.10.1}$$

the associated map (cf. 3.3.3). If moreover $\tilde{g}^P$ is strict or, equivalently, $(\tilde{g}^P)^*(\mathscr{P}_{B'}(P)) = \mathscr{P}_B(P)$ for any p-subgroup P of G, we say that $\tilde{g}$ is *strict*; notice that $\tilde{g}^P$ is strict if and only if we have $\mathrm{Ker}(g^P) \subset J(B^P)$. Recall that if $\tilde{g}^P$ is a semicovering exomorphism, $\gamma' \in \mathscr{P}_{B'}(P)$ and $\mathscr{S}(\gamma')$ is the set of $\gamma \in \mathscr{P}_B(P)$ such that $m(g)^\gamma_{\gamma'} \neq 0$ then $\tilde{g}^P$ induces an injective unitary exomorphism (cf. Prop. 3.3 and Remark 3.4)

$$\tilde{g}(P_{\gamma'}): \prod_{\gamma \in \mathscr{S}(\gamma')} B(P_\gamma) \to B'(P_{\gamma'}) . \tag{3.10.2}$$

First of all, we prove that the relative trace map preserves semicovering exomorphisms: this result is the key point to show in Theorem 3.16 below that semicovering exomorphisms of G-algebras can be recognized locally. Notice that the analogous statement on *covering exomorphisms* is true and supplies another approach to prove Theorem 4.22 of [18].

Theorem 3.11. *With the notation above, let* P *and* Q *be* p*-subgroups of* G *such that* $Q \subset P$, $\mathfrak{b}$ *an ideal of* B^Q *and* $\mathfrak{b}'$ *an ideal of* B'^Q *such that* $g(\mathfrak{b}) \subset \mathfrak{b}'$. *If the* $\mathcal{O}$*-algebra exomorphism* $\tilde{g}^Q$ *is a semicovering exomorphism from* $\mathfrak{b}$ *to* $\mathfrak{b}'$ *then* $\tilde{g}^P$ *is a semicovering exomorphism from* $\mathrm{Tr}_Q^P(\mathfrak{b})$ *to* $\mathrm{Tr}_Q^P(\mathfrak{b}')$.

Proof. We may assume that $G = P$ and $P \neq Q$. Arguing by induction on $|P:Q|$, if R is a maximal subgroup of P containing Q then $\tilde{g}^R$ is a semicovering exomorphism from $\mathrm{Tr}_Q^R(\mathfrak{b})$ to $\mathrm{Tr}_Q^R(\mathfrak{b}')$. Consequently, we may assume that $|P:Q| = p$, so that Q is

normal in P; then B^Q and B'^Q are (P/Q)-algebras and $\tilde{g}^Q$ is a semicovering exomorphism from $\sum_{u\in P} \mathfrak{b}^u$ to $\sum_{u\in P} \mathfrak{b}'^u$ (cf. Prop. 3.5). Since $\mathrm{Tr}_Q^P(\sum_{u\in P} \mathfrak{b}^u) = \mathrm{Tr}_Q^P(\mathfrak{b})$ and $\mathrm{Tr}_Q^P(\sum_{u\in P} \mathfrak{b}'^u) = \mathrm{Tr}_Q^P(\mathfrak{b}')$, we may assume that $Q = \{1\}$ and that P has order p and stabilizes $\mathfrak{b}$ and $\mathfrak{b}'$. Moreover, we may assume that $\mathcal{O} = \mathcal{k}$ (cf. 3.2).

Now $J(\mathfrak{b})$ and $J(\mathfrak{b}')$ are respectively P-stable ideals of B and B', and we have $g(J(\mathfrak{b})) \subset J(\mathfrak{b}')$ since the $\mathcal{O}$-algebra exomorphism $\tilde{g}$ is a semicovering from $\mathfrak{b}$ to $\mathfrak{b}'$ (cf. 3.2). Set $\bar{B} = B/J(\mathfrak{b})$ and $\bar{B}' = B'/J(\mathfrak{b}')$ and denote by $\bar{g}: \bar{B} \to \bar{B}'$ the P-algebra homomorphism induced by g, and by $\bar{\mathfrak{b}}$ and $\bar{\mathfrak{b}}'$ the respective images of $\mathfrak{b}$ in $\bar{B}$ and $\mathfrak{b}'$ in $\bar{B}'$; by Corollary 3.7, the $\mathcal{k}$-algebra exomorphism $\tilde{\bar{g}}$ is a semicovering from $\bar{\mathfrak{b}}$ to $\bar{\mathfrak{b}}'$. On the other hand, if the $\mathcal{k}$-algebra exomorphism $\tilde{\bar{g}}^P$ is a semicovering from $\mathrm{Tr}_1^P(\bar{\mathfrak{b}})$ to $\mathrm{Tr}_1^P(\bar{\mathfrak{b}}')$, the $\mathcal{k}$-algebra exomorphism from B^P to $\bar{B}'^P$ induced by $\tilde{\bar{g}}^P$ is a semicovering from $\mathrm{Tr}_1^P(\mathfrak{b})$ to $\mathrm{Tr}_1^P(\bar{\mathfrak{b}}')$ (cf. 3.6.1), and therefore it follows from 3.6.2 that $\tilde{g}^P$ is a semicovering exomorphism from $\mathrm{Tr}_1^P(\mathfrak{b})$ to $\mathrm{Tr}_1^P(\mathfrak{b}')$ (since $J(\mathrm{Tr}_1^P(\mathfrak{b}'))$ maps onto $J(\mathrm{Tr}_1^P(\bar{\mathfrak{b}}'))$). Hence we may assume that $J(\mathfrak{b}) = \{0\} = J(\mathfrak{b}')$. In that case, $\mathfrak{b}$ and $\mathfrak{b}'$ are actually semisimple $\mathcal{k}$-algebras and therefore we may assume that $\mathfrak{b} = B$ and $\mathfrak{b}' = B'$. Moreover, the exomorphism defined by the inclusion $g(B) \subset B'$ is clearly a semicovering and, since $g(B_1^P) = g(B)_1^P$, we may assume that $\tilde{g}$ is injective.

In conclusion, we assume that B and B' are semisimple and that $\tilde{g}$ is an injective semicovering exomorphism of $\mathcal{k}$-algebras, and then it suffices to prove that $\tilde{g}^P$ is a semicovering exomorphism from B_1^P to $B_1'^P$. Let I be the set of primitive idempotents of $Z(B)$; since $\tilde{g}$ is an injective semicovering, Bi and $g(i)B'g(i)$ are simple algebras for any $i \in I$ and therefore g induces a $\mathcal{k}$-algebra isomorphism $Bi \cong g(i)B'g(i)$. Moreover, by Propositions 3.5 and 3.8, we may assume that P acts transitively on I; then, if $|I| = 1$ or B' is not simple, g induces an isomorphism $B \cong B'$ and therefore an isomorphism $B_1^P \cong B_1'^P$. Otherwise, B' is simple and $|I| = p$; now, choosing $i \in I$, the trace map Tr_1^P induces a $\mathcal{k}$-algebra isomorphism $Bi \cong B^P = B_1^P$ and, lifting the action of P on B' to a group homomorphism $P \to (B')^\times$, we have $B' \cong \mathrm{Ind}_1^P(g(i)B'g(i))$ (cf. 2.13.2); in particular, $J(B^P) = \{0\}$ and it is easily checked that $B'^P \cong g(i)B'g(i) \otimes_{\mathcal{k}} \mathcal{k}P \cong B^P \otimes_{\mathcal{k}} \mathcal{k}P$ which proves that $\tilde{g}^P$ is a semicovering exomorphism.

Corollary 3.12. *Let P, Q and R be p-subgroups of G such that $R \subset Q \subset P$. If $\tilde{g}^Q$ is a semicovering exomorphism from B_R^Q to $B_R'^Q$ then $\tilde{g}^P$ is a semicovering exomorphism from B_R^P to $B_R'^P$.*

Proof. This follows from Theorem 3.11 applied to $\mathfrak{b} = B_R^Q$ and $\mathfrak{b}' = B_R'^Q$.

Now we generalize Propositions 3.6 and 3.8 to the new context.

Proposition 3.13. *With the notation above, let B'' be a G-algebra and $\tilde{h}: B' \to B''$ a G-algebra exomorphism.*

3.13.1. If $\tilde{g}$ and $\tilde{h}$ are semicovering exomorphisms, then $\tilde{h} \circ \tilde{g}$ is also a semicovering exomorphism and we have $((\tilde{h} \circ \tilde{g})^P)^* = (\tilde{g}^P)^* \circ (\tilde{h}^P)^*$ for any p-subgroup P of G. In particular, if two semicovering exomorphisms among $\tilde{g}$, $\tilde{h}$ and $\tilde{h} \circ \tilde{g}$ are strict, so is the third.

3.13.2. If $\tilde{h}\circ\tilde{g}$ is a semicovering exomorphism and, for any p-subgroup P of G, we have $h(J(B'^P))\subset J(B''^P)$ then $\tilde{h}$ is a semicovering exomorphism. If moreover $\tilde{h}$ is strict then $\tilde{g}$ is a semicovering exomorphism too.

Proof. This follows easily from Proposition 3.6.

Proposition 3.14. *With the notation above, let D and D' be G-algebras, $\tilde{e}\colon D\to B$ and $\tilde{e}'\colon D'\to B'$ G-algebra embeddings and $\tilde{h}\colon D\to D'$ a unitary G-algebra exomorphism such that $\tilde{e}'\circ\tilde{h}=\tilde{g}\circ\tilde{e}$. If $\tilde{g}$ is a semicovering exomorphism then $\tilde{h}$ is a semicovering exomorphism too. If moreover $\tilde{g}$ is strict then $\tilde{h}$ is strict too.*

Proof. This follows easily from Proposition 3.8.

The following proposition summarizes the lifting features of semicovering exomorphisms.

Proposition 3.15. *With the notation above, assume that $\tilde{g}$ is a semicovering exomorphism. Let P_γ and $Q_{\delta'}$ be respectively pointed p-groups on B and B' fulfilling $Q\subset P$, and denote by $\mathscr{S}(\delta')$ the set of $\delta\in\mathscr{P}_B(Q)$ such that $g(\delta)\subset\delta'$.*

3.15.1. If $Q_\delta\subset P_\gamma$ for some $\delta\in\mathscr{S}(\delta')$, there is $\gamma'\in\mathscr{P}_{B'}(P)$ such that $g(\gamma)\subset\gamma'$ and $Q_{\delta'}\subset P_{\gamma'}$.

3.15.2. If $g(\gamma)\subset\gamma'$ for some $\gamma'\in\mathscr{P}_{B'}(P)$ then $m_{\delta'}^{\gamma'}=\sum_{\delta\in\mathscr{S}(\delta')}m_\delta^\gamma$. In particular, if $Q_{\delta'}\subset P_{\gamma'}$ there is $\delta\in\mathscr{S}(\delta')$ such that $Q_\delta\subset P_\gamma$.

3.15.3. If $\delta\in\mathscr{S}(\delta')$ then Q_δ is local if and only if $Q_{\delta'}$ is local and, in particular, Q_δ is maximal local on B if $Q_{\delta'}$ is maximal local on B'.

Proof. Recall that (cf. [18], 2.3.1)

$$\sum_{\delta\in\mathscr{P}_B(Q)}m_\delta^\gamma m(g)_{\delta'}^\delta=m(g)_{\delta'}^\gamma=\sum_{\gamma'\in\mathscr{P}_{B'}(P)}m(g)_{\gamma'}^\gamma m_{\delta'}^{\gamma'}\,; \tag{3.15.4}$$

on the other hand, by 3.3.2 we have $m(g)_{\delta'}^\delta=1$ or 0 according to whether δ belongs to $\mathscr{S}(\delta')$ or not, and therefore if $Q_\delta\subset P_\gamma$ for some $\delta\in\mathscr{S}(\delta')$ we have $m(g)_{\delta'}^\gamma\neq 0$ (cf. 3.15.4) which implies $m(g)_{\gamma'}^\gamma\neq 0\neq m_{\delta'}^{\gamma'}$ for some $\gamma'\in\mathscr{P}_{B'}(P)$; hence, in that case, we have $Q_{\delta'}\subset P_{\gamma'}$ and, again by 3.3.2, $g(\gamma)\subset\gamma'$; moreover, if $g(\gamma)\subset\gamma'$ for some $\gamma'\in\mathscr{P}_{B'}(P)$, γ' is unique such that $m(g)_{\gamma'}^\gamma\neq 0$ and therefore (cf. 3.15.4)

$$\sum_{\delta\in\mathscr{S}(\delta')}m_\delta^\gamma=m(g)_{\delta'}^\gamma=m_{\delta'}^{\gamma'}\,. \tag{3.15.5}$$

If $\delta\in\mathscr{S}(\delta')$ is not local, there is a proper subgroup R of Q such that $\delta\subset B_R^Q$ whence $g(\delta)\subset B_R'^Q$, and $g(\delta)\subset\delta'$ forces $\delta'\subset B_R'^Q$; conversely, if $\delta'\subset B_R'^Q$ we have $\mathscr{S}(\delta')\subset\mathscr{P}(B_R^Q)$ since $\tilde{g}^Q$ is a semicovering exomorphism from B_R^Q to $B_R'^Q$ (cf. Cor. 3.12) and $(\tilde{g}^Q)^*$ extends the associated map from $\mathbb{S}(\mathscr{P}(B_R'^Q))$ to $\mathbb{S}(\mathscr{P}(B_R^Q))$ (cf. Prop. 3.5).

The following theorem may be considered as the main result of this section. It supplies "local" criteria to recognize semicovering exomorphisms. For any p-subgroup P of G, we denote by $\tilde{g}(P)\colon B(P)\to B'(P)$ the k-algebra exomorphism induced by $\tilde{g}$.

Theorem 3.16. *With the notation above, the following conditions on $\tilde{g}$ are equivalent*

3.16.1. The G-algebra exomorphism $\tilde{g}\colon B \to B'$ is a semicovering.

3.16.2. For any p-subgroup P of G, the $\mathcal{k}$-algebra exomorphism $\tilde{g}(P)\colon B(P) \to B'(P)$ is a semicovering.

3.16.3. For any local pointed group $P_{\gamma'}$ on B' there is $\mathcal{S}(\gamma') \subset \mathcal{P}_B(P)$ such that $m_{\gamma'} = \sum_{\gamma \in \mathcal{S}(\gamma')} m_\gamma$ and for any $\gamma \in \mathcal{S}(\gamma')$ we have

$$\mathrm{Br}_P(g(\gamma)) \subset \mathrm{Br}_P(\gamma') \qquad \text{and} \qquad \mathrm{Br}_P(g(J(B^P \cdot \gamma \cdot B^P))) \subset \mathrm{Br}_P(J(B'^P \cdot \gamma' \cdot B'^P)) .$$

3.16.4. For any p-subgroup P of G and any subgroup Q of P, the $\mathcal{O}$-algebra exomorphism $\tilde{g}^P\colon B^P \to B'^P$ is a semicovering from B_Q^P to $B_Q'^P$.

In that case, $\tilde{g}$ is strict if and only if $\tilde{g}(P)$ is strict for any p-subgroup P of G.

Proof. It is clear that 3.16.4 implies 3.16.1, and the converse follows from Theorem 3.11. On the other hand, it follows from Corollary 3.7 (applied to $\mathcal{k} \otimes \tilde{g}^P\colon \mathcal{k} \otimes_{\mathcal{O}} B^P \to \mathcal{k} \otimes_{\mathcal{O}} B'^P$ for any p-subgroup P of G) that 3.16.1 implies 3.16.2. Moreover, if $\tilde{g}$ is a strict semicovering, for any p-subgroup P of G we have $(\tilde{g}^P)^*(\mathcal{P}_{B'}(P)) = \mathcal{P}_B(P)$ (cf. Prop. 3.3) and, by Proposition 3.15, we get $(\tilde{g}^P)^*(\mathcal{L}\mathcal{P}_{B'}(P)) = \mathcal{L}\mathcal{P}_B(P)$ or, equivalently, $\tilde{g}(P)^*(\mathcal{P}(B'(P))) = \mathcal{P}(B(P))$, whence $\tilde{g}(P)$ is a strict semicovering too (cf. Cor. 3.7); the converse is clear.

It follows from Proposition 3.3, applied to $\tilde{g}(P)\colon B(P) \to B'(P)$ for any p-subgroup P of G, that 3.16.2 implies 3.16.3. Finally, assume that 3.16.3 holds and let P be a p-subgroup of G; we will prove by induction on $|P|$ that $\tilde{g}^P\colon B^P \to B'^P$ is a semicovering exomorphism. If Q is a proper subgroup of P, it follows from the induction hypothesis and Theorem 3.11 that $\tilde{g}^P$ is a semicovering exomorphism from B_Q^P to $B_Q'^P$. So, by Proposition 3.5, $\tilde{g}^P$ is a semicovering exomorphism from $\mathrm{Ker}(\mathrm{Br}_P^B)$ to $\mathrm{Ker}(\mathrm{Br}_P^{B'})$ (notice that this is trivially true if $P = \{1\}$); in particular, for any $\gamma \in \mathcal{L}\mathcal{P}_B(P)$ and any $\gamma' \in \mathcal{P}_{B'}(P) - \mathcal{L}\mathcal{P}_{B'}(P)$ we have $m(g)_{\gamma'}^{\gamma} = 0$ (cf. Remark 3.4). Moreover, for any $\gamma' \in \mathcal{L}\mathcal{P}_{B'}(P)$ there is $\mathcal{S}(\gamma') \subset \mathcal{P}_B(P)$ such that $m_{\gamma'} = \sum_{\gamma \in \mathcal{S}(\gamma')} m_\gamma$ and such that for any $\gamma \in \mathcal{S}(\gamma')$ we have

$$\mathrm{Br}_P(g(\gamma)) \subset \mathrm{Br}_P(\gamma') \qquad \text{and} \qquad \mathrm{Br}_P(g(J(B^P \cdot \gamma \cdot B^P))) \subset \mathrm{Br}_P(J(B'^P \cdot \gamma' \cdot B'^P)) ,$$

which implies that $g(\gamma) \subset \gamma'$ (since $m(g)_{\delta'}^{\gamma} = 0$ for any $\delta' \in \mathcal{P}(\mathrm{Ker}(\mathrm{Br}_P^{B'}))$) and therefore that $g(J(B^P \cdot \gamma \cdot B^P)) \subset J(B'^P \cdot \gamma' \cdot B'^P)$ (since $g(J(B^P \cdot \gamma \cdot B^P))$ is now contained in $B'^P \cdot \gamma' \cdot B'^P$ and maps onto $\{0\}$ in $B'(P_\gamma)$); hence, as $J(\sum_{\gamma \in \mathcal{S}(\gamma')} B^P \cdot \gamma \cdot B^P) = \sum_{\gamma \in \mathcal{S}(\gamma')} J(B^P \cdot \gamma \cdot B^P)$, $\tilde{g}^P$ is a semicovering exomorphism from $\sum_{\gamma \in \mathcal{S}(\gamma')} B^P \cdot \gamma \cdot B^P$ to $B'^P \cdot \gamma' \cdot B'^P$ (cf. 3.3.2). Finally, if $\gamma \in \mathcal{L}\mathcal{P}_B(P) - \bigcup_{\gamma'} \mathcal{S}(\gamma')$, where γ' runs over $\mathcal{L}\mathcal{P}_{B'}(P)$, we have $m(g)_{\gamma'}^{\gamma} = 0$ for any $\gamma' \in \mathcal{P}_{B'}(P)$, and therefore $g(\gamma) = \{0\}$; thus $\tilde{g}^P$ is also a semicovering exomorphism from $B^P \cdot \gamma \cdot B^P$ to $\{0\}$. Consequently, as

$$B^P = \sum_{\gamma \in \mathcal{L}\mathcal{P}_B(P)} B^P \cdot \gamma \cdot B^P + \mathrm{Ker}(\mathrm{Br}_P^B) \text{ and } B'^P = \sum_{\gamma' \in \mathcal{L}\mathcal{P}_{B'}(P)} B'^P \cdot \gamma' \cdot B'^P + \mathrm{Ker}(\mathrm{Br}_P^{B'}) ,$$

the statement follows now from Proposition 3.5.

4. Auxiliary results on group extensions

4.1. Let L, L' and F be groups and $\rho: L \to F$ and $\rho': L' \to F$ surjective group homomorphisms; set $Z = \mathrm{Ker}(\rho)$ and $Z' = \mathrm{Ker}(\rho')$. Assume that Z and Z' are p'-divisible abelian groups, so that they become $\mathbf{Z}_{(p)}F$-modules through the respective actions of L and L'. Moreover, assume that L and therefore F and Z are finite, denote by P a Sylow p-subgroup of L, so that $Z \subset P$, and set $\rho(P) = \bar{P}$. As usual, we consider respectively L and L' endowed with ρ and ρ' as group extensions of F by the $\mathbf{Z}_{(p)}F$-modules Z and Z'.

4.2. It is well-known from homological algebra (cf. [4], Ch. XIV, Th. 3.1 and 4.2) that L and L' determine respectively elements $h \in \mathbb{H}^2(F, Z)$ and $h' \in \mathbb{H}^2(F, Z')$ (cf. 2.19), and that there is a group homomorphism $\sigma: L \to L'$ lifting id_F and extending a $\mathbf{Z}_{(p)}F$-module homomorphism $\theta: Z \to Z'$ if and only if the homomorphism $\mathbb{H}^2(F, \theta)$ maps h upon h'. Moreover, as restriction induces an injective map $\mathbb{H}^2(F, Z) \to \mathbb{H}^2(\bar{P}, Z)$ (cf. 2.19), there is such a group homomorphism $\sigma: L \to L'$ if and only if there is a group homomorphism $\tau: P \to L'$ lifting the inclusion $\bar{P} \subset F$ and extending θ. But, in this case, σ does *not necessarily* extend τ or, more generally, any L'-conjugate of τ; indeed, the "difference" between τ and the restriction of σ to P determines an element of $\mathbb{H}^1(\bar{P}, Z')$ (see the proof of Proposition 4.4 below) which is not necessarily the zero element. In this section we discuss a sufficient group-theoretical condition to guarantee that σ extends τ up to L'-conjugation, which is still useful for non-abelian extensions (see Proposition 4.9 below).

4.3. We consider F endowed with the group homomorphism $F \to \mathrm{Aut}(Z')$ induced by ρ' as a Z'-*permutation group* (cf. 2.4). Recall that, considering the trivial F-algebra $\mathcal{O}$ (and omitting to mention the points), an (F, Z')-*biexomorphism* $(\varphi, \psi) = (\bar{R}, \bar{x}, \bar{y}, \bar{P})_{Z'}$ (cf. 2.15) is a pair of elements of $E_{F, Z'}(\bar{R}, \bar{P})$ (cf. 2.8) where $\bar{R}$ is a p-subgroup of F and $\bar{x}, \bar{y} \in F$ fulfil $\bar{R}^{\bar{x}} \subset \bar{P}$ and $\bar{R}^{\bar{y}} \subset \bar{P}$; moreover, a *generator set* of the (F, Z')-biexomorphisms is a set $\mathfrak{C}$ of (F, Z')-biexomorphisms such that any (F, Z')-biexomorphism is a linear combination of $\mathfrak{C}$ (cf. 2.15.3 and 2.16). Notice that, by 2.16.1, 2.17.1 and 2.18.4, the family $\{(\bar{P} \cap \bar{P}^{\bar{x}^{-1}}, 1, \bar{x}, \bar{P})_{Z'}\}_{\bar{x} \in F}$ is a generator set of the (F, Z')-biexomorphisms.

Proposition 4.4. *Let $\bar{X}$ be a subset of F containing $N_F(\bar{P})$ such that the family $\{(\bar{P} \cap \bar{P}^{\bar{x}^{-1}}, 1, \bar{x}, \bar{P})_{Z'}\}_{\bar{x} \in \bar{X}}$ is a generator set of the (F, Z')-biexomorphisms. Let $\tau: P \to L'$ be a group homomorphism fulfilling the following conditions:*

4.4.1. We have $\rho'(\tau(u)) = \rho(u)$ for any $u \in P$, and $\tau(z^{\bar{x}}) = \tau(z)^{\bar{x}}$ for any $z \in Z$ and any $\bar{x} \in F$.

4.4.2. If $\bar{x} \in \bar{X}$ there are $y \in L$ and $y' \in L'$ such that $\rho(y) = \bar{z}\bar{x} = \rho'(y')$ for some $\bar{z} \in C_F(\bar{P} \cap \bar{P}^{\bar{x}^{-1}}, Z')$ and $\tau(u^y) = \tau(u)^{y'}$ for any $u \in \rho^{-1}(\bar{P} \cap \bar{P}^{\bar{x}^{-1}})$.

Then there is a unique group exomorphism $\tilde{\sigma}: L \to L'$ having a representative σ such that $\rho' \circ \sigma = \rho$ and $\sigma(u) = \tau(u)$ for any $u \in P$.

Remark 4.5. Conditions 4.4.1 and 4.4.2 are obviously necessary.

Remark 4.6. Notice that if $\varphi, \psi \in \mathrm{Hom}(L, L')$ fulfil the equalities $\rho' \circ \varphi = \rho' \circ \psi$ and $\varphi(u) = \psi(u)$ for any $u \in P$ then there is $z' \in C_{Z'}(\varphi(P))$ such that $\psi(x) = \varphi(x)^{z'}$ for any $x \in L$. Indeed, the 1-cocycle mapping $x \in L$ upon $\psi(x)^{-1}\varphi(x) \in Z'$ determines the zero element of $\mathrm{H}^1(L, \mathrm{Res}_{\rho' \circ \varphi}(Z'))$ since its restriction to P is the trivial homomorphism from P to Z' (cf. 2.19); hence there is $z' \in Z'$ such that

$$\psi(x)^{-1}\varphi(x) = z'^{-1}z'^{x} = z'^{-1}\varphi(x)^{-1}z'\varphi(x)$$

for any $x \in L$, whence $\psi(x) = \varphi(x)^{z'}$.

Proof. Denoting by $\theta: Z \to Z'$ the $\mathbf{Z}_{(p)}F$-module homomorphism induced by τ and arguing as in 4.2, we get a group homomorphism $\varphi: L \to L'$ such that $\rho' \circ \varphi = \rho$ and $\varphi(z) = \tau(z)$ for any $z \in Z$; in particular, we have $\rho'(\varphi(u)) = \rho(u) = \rho'(\tau(u))$ for any $u \in P$, and therefore $\tau(u) = \varphi(u)\zeta(u)$ where the map $\zeta: P \to Z'$ is a 1-cocycle; moreover, as $\zeta(z) = 1$ for any $z \in Z$, ζ induces 1-cocycle $\bar{\zeta}: \bar{P} \to Z'$ which determines an element h of $\mathrm{H}^1(\bar{P}, Z')$. Now it suffices to prove that h is the restriction of an element of $\mathrm{H}^1(F, Z')$; indeed, in this case it is quite clear that there is a 1-cocycle $\bar{\xi}: F \to Z'$ extending $\bar{\zeta}$, and then it is easily checked that the map $\sigma: L \to L'$ assigning to $x \in L$ the product $\varphi(x)\bar{\xi}(\rho(x)) \in L'$ is a group homomorphism extending τ and fulfilling $\rho' \circ \sigma = \rho$. Moreover, the uniqueness of $\tilde{\sigma}$ follows from Remark 4.6.

To show that h belongs to the image of $\mathrm{H}^1(F, Z')$, it suffices to prove that, for any $\bar{x} \in \bar{X}$, the 1-cocycle $\bar{P} \cap \bar{P}^{\bar{x}^{-1}} \to Z'$ mapping $\bar{u} \in \bar{P} \cap \bar{P}^{\bar{x}^{-1}}$ upon $\bar{\zeta}(\bar{u})^{-1}\bar{\zeta}(\bar{u}^{\bar{x}})^{\bar{x}^{-1}} \in Z'$ is a 1-coboundary (cf. 2.19.2). By condition 4.4.2, for any $\bar{x} \in \bar{X}$ there are $y \in L$ and $y' \in L'$ fulfilling $\rho(y) = \overline{zx} = \rho'(y')$ for some $\bar{z} \in C_F(\bar{P} \cap \bar{P}^{\bar{x}^{-1}}, Z')$ and $\tau(u^y) = \tau(u)^{y'}$ for any $u \in \rho^{-1}(\bar{P} \cap \bar{P}^{\bar{x}^{-1}})$; now, setting $z' = \varphi(y)y'^{-1}$, we have

$$\varphi(u)\zeta(u) = \tau(u) = \tau(u^y)^{y'^{-1}} = \varphi(u^y)^{y'^{-1}}\zeta(u^y)^{y'^{-1}} = \varphi(u)^{z'}\zeta(u^y)^{y'^{-1}}$$

and therefore

$$\bar{\zeta}(\bar{u})^{-1}\bar{\zeta}(\bar{u}^{\bar{x}})^{\bar{x}^{-1}} = \zeta(u)^{-1}\zeta(u^y)^{y'^{-1}} = z'^{-1}z'^{u}$$

for any $\bar{u} \in \bar{P} \cap \bar{P}^{\bar{x}^{-1}}$, where $u \in P$ lifts $\bar{u}$. Consequently, the 1-cocycle above is indeed a 1-coboundary.

In particular, Proposition 4.4 implies the following splitting criterion.

Corollary 4.7. *With the notation of Proposition* 4.4., *let* $\bar{\tau}: \bar{P} \to L'$ *be a group homomorphism fulfilling the following conditions*:

4.7.1. We have $\rho'(\bar{\tau}(\bar{u})) = \bar{u}$ *for any* $\bar{u} \in \bar{P}$.

4.7.2. If $\bar{x} \in \bar{X}$ *there is* $x' \in L'$ *such that* $\rho'(x') = \bar{x}$ *and* $\bar{\tau}(\bar{u}^{\bar{x}}) = \bar{\tau}(\bar{u})^{x'}$ *for any* $\bar{u} \in \bar{P} \cap \bar{P}^{\bar{x}^{-1}}$.

Then there is a unique group exomorphism $\bar{\tilde{\sigma}}: F \to L'$ *having a representative* $\bar{\sigma}$ *such that* $\rho' \circ \bar{\sigma} = \mathrm{id}_F$ *and* $\bar{\sigma}(\bar{u}) = \bar{\tau}(\bar{u})$ *for any* $\bar{u} \in \bar{P}$.

Proof. This follows trivially from Proposition 4.4 setting $L = F$ and $\rho = \mathrm{id}_F$.

4.8. Let E be a finite group and $\pi: L \to E$ and $\pi': L' \to E$ surjective group homomorphisms. Assume that π and π' have the same kernel Q and induce the

same group exomorphism from E to $\tilde{\mathrm{A}}\mathrm{ut}(Q)$; that is, denoting respectively by η and η' the group homomorphisms from L and L' to $\mathrm{Aut}(Q)$ induced by conjugation, and by $\bar{\eta}$ and $\bar{\eta}'$ the corresponding homomorphisms from E to $\tilde{\mathrm{A}}\mathrm{ut}(Q)$, we assume that $\tilde{\bar{\eta}} = \tilde{\bar{\eta}}'$. Moreover, assume that Q is a p-group, so that $Q \subset P$, and consider L as an *$\varnothing$-permutation group* (cf. 2.4) (omitting to mention the empty set).

Proposition 4.9. *Let X be a subset of L containing $N_L(P)$ such that the family $\{(P \cap P^{x^{-1}}, 1, x, P)\}_{x \in X}$ is a generator set of the L-biexomorphisms. Let $\tau: P \to L'$ be an injective group homomorphism fulfilling the following conditions:*

4.9.1. We have $\pi'(\tau(u)) = \pi(u)$ for any $u \in P$, and $\bar{\eta}(e) = \bar{\eta}'(e)^{\tilde{\tau}_0}$ for any $e \in E$, where τ_0 is the automorphism of Q induced by τ.

4.9.2. If $x \in X$ there is $x' \in L'$ such that $\pi'(x') = \pi(zx)$ for some $z \in C_L(P \cap P^{x^{-1}})$ and $\tau(u^x) = \tau(u)^{x'}$ for any $u \in P \cap P^{x^{-1}}$.

Then there is a unique group exoisomorphism $\tilde{\sigma}: L \cong L'$ having a representative σ such that $\pi' \circ \sigma = \pi$ and $\sigma(u) = \tau(u)$, for any $u \in P$.

Remark 4.10. Conditions 4.9.1 and 4.9.2 are obviously necessary.

Remark 4.11. Considering respectively L and L' endowed with π and π', and with left multiplication as *E-permutation groups* (cf. 2.4), condition 4.9.1 implies that τ induces an evident bijection $\tau_R: \tilde{\mathrm{H}}\mathrm{om}_E(R, P) \cong \tilde{\mathrm{H}}\mathrm{om}_E(\tau(R), \tau(P))$ for any subgroup R of P, and then condition 4.9.2 can be restated as follows:

4.11.1. If $x \in X$ and $R = P \cap P^{x^{-1}}$, there is $\varphi \in E_{L,E}(R, P)$ lifting $(R, x, P) \in E_L(R, P)$ such that $\tau_R(\varphi) \in E_{L',E}(\tau(R), \tau(P))$.

Indeed, if 4.9.2 holds it suffices to set $\varphi = (R, zx, P)_E$. Conversely, if $\varphi = (R, y, P)_E$ lifts (R, x, P) we have $y = zxv$ for some $z \in C_L(R)$ and $v \in P$ (cf. 2.8). and if $\tau_R(\varphi) \in E_{L',E}(\tau(R), \tau(P))$ there is $y' \in L'$ such that $\pi'(y') = \pi(y)$ and $\tau(u^y) = \tau(u)^{y'}$ for any $u \in P$ (cf. 2.8), whence $\pi'(y'\tau(v)^{-1}) = \pi(zx)$ and $\tau(u^x) = \tau(u)^{y'\tau(v)^{-1}}$ for any $u \in P$.

Proof. We may assume that F above is the pull-back of $\bar{\eta}: E \to \tilde{\mathrm{A}}\mathrm{ut}(Q)$ and the canonical homomorphism $\mathrm{Aut}(Q) \to \tilde{\mathrm{A}}\mathrm{ut}(Q)$ (i.e. the subgroup of pairs $(e, \theta) \in E \times \mathrm{Aut}(Q)$ such that $\bar{\eta}(e) = \tilde{\theta}$), and that ρ and ρ' are respectively the group homomorphisms mapping $x \in L$ upon $(\pi(x), \eta(x))$ and $x' \in L'$ upon $(\pi'(x'), \eta'(x')^{\tau_0})$. Then, it is clear that $Z = Z(Q) = Z'$; moreover, notice that ρ and ρ' induce the same group homomorphism $F \to \mathrm{Aut}(Z)$ since $\widetilde{\eta(x)} = \widetilde{\eta'(x')}^{\tilde{\tau}_0}$, for any $x \in L$ and any $x' \in L'$ such that $\pi(x) = \pi'(x')$, by condition 4.9.1.

Now, considering F endowed with this group homomorphism as a *Z-permutation group* (cf. 2.4), it is still clear that ρ induces a surjective map $E_L(R, P) \to E_{\bar{F},Z}(\rho(R), \rho(P))$ for any p-subgroup R of L containing Q; in particular, setting $\bar{X} = \rho(X)$ and $\bar{P} = \rho(P)$, the family $\{(\bar{P} \cap \bar{P}^{\bar{x}^{-1}}, 1, \bar{x}, \bar{P})_Z\}_{\bar{x} \in \bar{X}}$ is a generator set of the (F, Z)-biexomorphisms (since any (F, Z)-biexomorphism can be lifted to an L-biexomorphism). Therefore it suffices to prove that conditions 4.4.1 and 4.4.2 (with respect to $\bar{X}$ and τ) follow from conditions 4.9.1 and 4.9.2. Indeed, it follows then from Proposition 4.4 that there is a group homomorphism

$\sigma: L \to L'$ which extends τ and fulfils $\rho' \circ \sigma = \rho$, whence $\pi' \circ \sigma = \pi$; moreover, σ is an isomorphism since τ is injective and $|L| = |L'|$, and the uniqueness of $\tilde{\sigma}$ follows from Remark 4.12 below (or directly from Proposition 4.4).

First of all, we prove that 4.9.1 implies 4.4.1; as $(\eta'(\tau(u))^{\tau_0})(v) = v^u = \eta(u)(v)$ for any $u \in P$ and any $v \in Q$, it is clear that $\pi'(\tau(u)) = \pi(u)$ implies $\rho'(\tau(u)) = \rho(u)$ for any $u \in P$; moreover, if $x \in L$ and $x' \in L'$ fulfil $\rho(x) = \rho'(x')$ then $\tau(\eta(x)(z)) = \eta'(x')(\tau(z))$, whence $\tau(z^x) = \tau(z)^{x'}$ for any $z \in Z$. Secondly, by condition 4.9.2, if $\bar{x} \in \bar{X}$ and $x \in X$ lifts $\bar{x}$ there is $x' \in L'$ such that $\pi'(x') = \pi(zx)$ for some $z \in C_L(P \cap P^{x^{-1}})$ and $\tau(u^{zx}) = \tau(u)^{x'}$ for any $u \in P \cap P^{x^{-1}}$, and it suffices to set $y = zx$ and $y' = x'$ in 4.4.2; indeed, as $\tau(\eta(y)(v)) = \eta'(y')(\tau(v))$ for any $v \in Q$ (since $Q \subset P \cap P^{x^{-1}}$), we have $\rho'(y') = \rho(y) = \bar{z}\bar{x}$ where $\bar{z} = \rho(z)$ belongs to $C_F(\bar{P} \cap \bar{P}^{\bar{x}^{-1}}, Z)$ (once again since $Q \subset P \cap P^{x^{-1}}$), and $\tau(u^y) = \tau(u)^{y'}$ for any $u \in P \cap P^{x^{-1}} = \rho^{-1}(\bar{P} \cap \bar{P}^{\bar{x}^{-1}})$.

Remark 4.12. If φ and ψ are injective group homomorphisms from L to L' fulfilling the equalities $\pi' \circ \varphi = \pi' \circ \psi$ and $\varphi(u) = \psi(u)$ for any $u \in P$, there is $z \in C_Q(P)$ such that $\psi(x) = \varphi(x^z)$ for any $x \in L$. Indeed, with the notation above where τ is the restriction of φ and ψ to P) it is easily checked that $\rho' \circ \varphi = \rho' \circ \psi$, and it suffices to apply Remark 4.6.

5. On the local structure of block extensions

5.1. Let G be a finite group and H a normal subgroup of G, set $A = \mathcal{O}G$ and $B = \mathcal{O}H$, and consider A endowed with B as a *G-algebra extension* (cf. 2.9). Let α and β be respectively points of G and H on B (cf. 2.7) such that $H_\beta \subset G_\alpha$, set $N = N_G(H_\beta)$ and $\bar{N} = N/H$, and denote by $\bar{X}$ the image of $X \subset N$ in $\bar{N}$. In this section we will discuss to what extent the *local category* of B_α (cf. 2.8 and 2.11) is determined by the quotient $\bar{N}$. Actually we are only concerned with the case where the *local structure* of H_β is *nilpotent* (cf. [18], 1.7) – what we assume from 5.6 on – which is completely handled in Theorem 5.8 below. However it is not difficult to discuss the general situation from our results, as we explain in Remark 5.9 below.

5.2. Obviously β has just one element b and $\alpha = \{\mathrm{Tr}_N^G(b)\}$; hence β is still a point of N on B, therefore we get $A_\alpha \cong \mathrm{Ind}_N^G(A_\beta)$ (cf. 2.13.2); in particular, it is easily checked that the inclusion $N \subset G$ induces an equivalence between the *local categories* of B_α and B_β (cf. 2.8). That is, for our purposes it suffices to consider the local pointed groups on B contained in N_β, and the next statements show that the set of their images in $\bar{N}$ is the full set of p-subgroups.

Proposition 5.3. *A local pointed group P_γ on B such that $P_\gamma \subset N_\beta$ is a defect pointed group of G_α if and only if $\bar{P}$ is a Sylow p-subgroup of $\bar{N}$ and there is a defect pointed group Q_δ of H_β such that $Q_\delta \subset P_\gamma$, and then we have $Q = H \cap P$.*

Remark 5.4. Notice that Proposition 5.3 depends only on the G-algebra structure of B and therefore applies to each subgroup of G containing H.

Proof. Assume first that P_γ is a defect pointed group of G_α; in particular, P_γ is also a defect pointed group of N_β, and therefore a defect pointed group Q_δ of H_β has an

N-conjugate contained in P_γ; so we may assume that $Q_\delta \subset P_\gamma$. Moreover, as H is a P-stable $\mathcal{O}$-basis of B, there is $\varepsilon \in \mathcal{LP}_B(P \cap H)$ such that $(P \cap H)_\varepsilon \subset P_\gamma$ (cf. [1], Prop. 1.5); hence $(P \cap H)_\varepsilon \subset N_\beta$ and therefore $(P \cap H)_\varepsilon \subset H_\beta$ (since $s_\varepsilon(b) \neq 0$); thus there is $h \in H$ such that $(P \cap H)_\varepsilon \subset (Q_\delta)^h$ which forces $P \cap H = Q$ (since $Q \subset P \cap H$). On the other hand, β is still a point of $P \cdot H$ on B and P_γ a defect pointed group of $P \cdot H_\beta$; therefore, as $N_G(P_\gamma) \subset N$ (since $b\gamma = \gamma$), the Frattini argument shows that $\overline{N_G(P_\gamma)} = N_{\bar{N}}(\bar{P})$, so that $N_G(P_\gamma)/P \cdot C_H(P)$ maps surjectively on $N_{\bar{N}}(\bar{P})/\bar{P}$; but the quotient $N_G(P_\gamma)/P \cdot C_H(P)$ is a p'-group (cf. 2.16.2); hence $\bar{P}$ is a Sylow p-subgroup of $\bar{N}$.

Conversely, assume that $\bar{P}$ is a Sylow p-subgroup of $\bar{N}$ and that there is a defect pointed group Q_δ of H_β such that $Q_\delta \subset P_\gamma$; now, if R_ε is a defect pointed group of N_β (and therefore of G_α) such that $P_\gamma \subset R_\varepsilon$, we have $\bar{P} = \bar{R}$ and, by the argument above, $Q = R \cap H$ so that $P \cap H = R \cap H$; it follows that $P = R$, and so $P_\gamma = R_\varepsilon$.

Proposition 5.5 *Let P_γ and Q_δ be respectively defect pointed groups of G_α and H_β such that $Q_\delta \subset P_\gamma$. For any subgroup R of P containing Q there is a unique $\varepsilon \in \mathcal{LP}_B(R)$ such that $R_\varepsilon \subset P_\gamma$, and then it fulfils the following conditions:*

5.5.1. We have $Q_\delta \subset R_\varepsilon$, $N_G(R_\varepsilon) \subset N_G(Q_\delta)$ and $N_G(R_\varepsilon) \cap C_H(Q) = Z(Q) \cdot C_H(R)$.

5.5.2. Let L_λ be a pointed group on B such that $R_\varepsilon \subset L_\lambda$ and $C_H(R) \subset L \subset N_G(R_\varepsilon)$. A local pointed group T_ψ on B such that $R_\varepsilon \subset T_\psi \subset L_\lambda$ is a defect pointed group of L_λ if and only if $\bar{T}$ is a Sylow p-subgroup of $\bar{L}$.

Proof. First of all, notice that $P_\gamma \subset N_\beta$; indeed, anyway there is $x \in G$ such that $(P_\gamma)^x \subset N_\beta$ (since a defect pointed group of N_β is a defect pointed group of G_α), and then there is $h \in H$ such that $(Q_\delta)^{xh} = Q_\delta$ (since $(Q_\delta)^x \subset H_\beta$), whence $x \in N$ (since $N_G(Q_\delta) \subset N$). Moreover, as H is a P-stable $\mathcal{O}$-basis of B, there is $\varepsilon \in \mathcal{LP}_B(R)$ such that $R_\varepsilon \subset P_\gamma$, and if $\varepsilon' \in \mathcal{LP}_B(R)$ there is also $\delta' \in \mathcal{LP}_B(Q)$ such that $Q_{\delta'} \subset R_{\varepsilon'}$ (cf. [1], Prop. 1.5); now, assuming that $R_{\varepsilon'} \subset P_\gamma$ we get $Q_{\delta'} \subset N_\beta$ and therefore $Q_{\delta'} \subset H_\beta$ (since $b\delta' = \delta'$).

Let L_λ and T_ψ be pointed groups on B such that T_ψ is local, $R_\varepsilon \subset T_\psi \subset L_\lambda$ and $C_H(R) \subset L \subset N_G(R_\varepsilon)$, and set $\bar{\lambda} = \mathrm{Br}_R(\lambda)$, $\bar{\psi} = \mathrm{Br}_R(\psi)$ and $\bar{\varepsilon} = \mathrm{Br}_R(\varepsilon)$. As H is an L-stable $\mathcal{O}$-basis of B, it is easily checked that $\mathrm{Br}_R(B^L) = B(R)^L$ and $\mathrm{Br}_R(B^T) = B(R)^T$, and therefore $\bar{\lambda}$, $\bar{\psi}$ and $\bar{\varepsilon}$ are respectively points of L, T and R on $B(R)$ fulfilling $R_{\bar{\varepsilon}} \subset T_{\bar{\psi}} \subset L_{\bar{\lambda}}$; moreover, as $B(R) \cong \mathfrak{k}C_H(R)$ (cf. [19], 2.9.2) and $\bar{\varepsilon} \in \mathcal{P}(B(R))$, there is a block f of $C_H(R)$ such that $\bar{\lambda} = \{\bar{f}\}$ where $\bar{f} = \mathrm{Br}_R(f)$ (since $\overline{f\varepsilon} = \bar{\varepsilon}$ and L normalizes R_ε), and therefore $\bar{\lambda}$ is still a point of $C_H(R)$, $N_G(R_\varepsilon) \cap C_H(Q)$ and $R \cdot C_H(R)$ on $B(R)$.

First of all, it follows from Proposition 5.3 applied to $L_{\bar{\lambda}}$, $C_H(R)_{\bar{\lambda}}$ and $T_{\bar{\psi}}$ that if $T_{\bar{\psi}}$ is a defect pointed group of $L_{\bar{\lambda}}$ then $\bar{T}$ is a Sylow p-subgroup of $\bar{L}$ (and it is clear that if T_ψ is a defect pointed group of L_λ then $T_{\bar{\psi}}$ is a defect pointed group of $L_{\bar{\lambda}}$); conversely, again by Proposition 5.3, if $\bar{T}$ is a Sylow p-subgroup of $\bar{L}$ then T_ψ is a defect pointed group of $L \cdot H_\beta$ (cf. Remark 5.4) since $Q_{\delta''} \subset T_\psi$ for some $\delta'' \in \mathcal{LP}_B(Q)$ (and then $Q_{\delta''}$ is a defect pointed group of H_β), and therefore, as $L_\lambda \subset L \cdot H_\beta$, T_ψ is *a fortiori* a defect pointed group of L_λ. In particular, $R_{\bar{\varepsilon}}$ is a defect

pointed group of $R \cdot C_H(R)_{\bar{\lambda}}$, and therefore the block f of $R \cdot C_H(R)$ has a *nilpotent local structure* (cf. [18], 1.7); but, as $\overline{f\varepsilon} = \bar{\varepsilon}$ and P_γ contains both R_ε and $R_{\varepsilon'}$, we have also $\overline{f\varepsilon'} = \bar{\varepsilon}'$ where $\bar{\varepsilon}' = \mathrm{Br}_R(\varepsilon')$ (cf. [1], Th.1.8); consequently, we have $\bar{\varepsilon} = \bar{\varepsilon}'$ (cf. [2], Th. 1.2 or [18], Th. 1.6), whence $\varepsilon = \varepsilon'$. It is now clear that $\delta' = \delta$ (since $Q_{\delta'} \subset P_\gamma$) and that $N_G(R_\varepsilon) \subset N_G(Q_\delta)$ (since $Q = R \cap H$).

Finally, set $\bar{\beta} = \mathrm{Br}_R(\beta)$; as $\mathrm{Br}_R(B_R^{R \cdot H}) = B(R)_R^{N_{R \cdot H}(R)}$ (cf. [1], Lemma 1.12[6]) and R_ε is a defect pointed group of $R \cdot H_\beta$ (cf. Prop. 5.3 and Remark 5.4), $\bar{\beta}$ is a point of $N_{R \cdot H}(R)$ on $B(R)$ and $R_{\bar{\varepsilon}}$ is a defect pointed group of $N_{R \cdot H}(R)_{\bar{\beta}}$. On the other hand, $\bar{\varepsilon}$ is also a local point of $Z(Q)$ on $B(R)$ since $\bar{\varepsilon} \in \mathscr{P}(B(R))$, and setting $M = N_G(R_\varepsilon) \cap C_H(Q)$ we have $Z(Q)_{\bar{\varepsilon}} \subset M_{\bar{\lambda}}$ since $\overline{f\varepsilon} = \bar{\varepsilon}$. Now, as $Z(Q)_{\bar{\varepsilon}} \subset R_{\bar{\varepsilon}}$, $M_{\bar{\lambda}} \subset N_{R \cdot H}(R)_{\bar{\beta}}$ and $R \cap M = Z(Q)$, it is quite clear that $Z(Q)_{\bar{\varepsilon}}$ is a (the unique) defect pointed group of $M_{\bar{\lambda}}$, and it follows from Proposition 5.3 (applied to $M_{\bar{\lambda}}$, $C_H(R)_{\bar{\lambda}}$ and $Z(Q)_{\bar{\varepsilon}}$) that $M/Z(Q) \cdot C_H(R)$ is a p'-group; but for any p'-element s of M we have $[s, R] \subset R \cap H = Q$ and $[s, Q] = 1$, whence s centralizes R (cf. [9], K . I, 4.4); so $M/Z(Q) \cdot C_H(R) = \{1\}$.

5.6. From now on we assume that b is a block of H with *nilpotent local structure* (cf. [18], 1.7). Let P_γ and Q_δ be respectively defect pointed groups of G_α and H_β such that $Q_\delta \subset P_\gamma$, and R_ε a local pointed group on B such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$. As $N_H(Q_\delta) = Q \cdot C_H(Q)$ (cf. [18], 1.7.2), on one hand it follows from 5.5.1 that

$$N_H(R_\varepsilon) = Q \cdot C_H(R) . \tag{5.6.1}$$

On the other hand, as $N = H \cdot N_G(Q_\delta)$ by the Frattini argument, the inclusion $N_G(Q_\delta) \subset N$ induces a group isomorphism $\bar{N} \cong N_G(Q_\delta)/Q \cdot C_H(Q)$ (cf. 1.7.2); then this isomorphism and the group homomorphism $\eta\colon N_G(Q_\delta) \to \mathrm{Aut}(Q)$ induced by conjugation determine a group homomorphism

$$\bar{\eta}\colon \bar{N} \to \tilde{\mathrm{A}}\mathrm{ut}(Q) \tag{5.6.2}$$

and we set $\tilde{\bar{N}} = \bar{N}/\mathrm{Ker}(\bar{\eta})$ and consider $\bar{N}$, endowed with the action on $\tilde{\bar{N}}$ induced by left multiplication, as an $\tilde{\bar{N}}$-*permutation group* (cf. 2.4); notice that the group exomorphism $\tilde{\bar{\eta}}$ does not depend on the choice of δ, and therefore $\tilde{\bar{N}}$ does not depend on the choice of Q_δ.

5.7. Now consider the trivial $\bar{N}$-algebra $\mathcal{O}$ (omitting to mention the points) and the full subcategory of the *local category* of the N-algebra B_β (cf. 2.8) formed by the local pointed groups on B which contain a defect pointed group of H_β; the canonical homomorphism $N \to \bar{N}$ induces an evident functor from this subcategory to the *local category* of the $(\bar{N}, \tilde{\bar{N}})$-algebra $\mathcal{O}$ (cf. 2.6, 2.7 and 2.8); precisely, for any local pointed group T_ψ on B such that $Q_\delta \subset T_\psi \subset P_\gamma$, we have the map

$$E_N(R_\varepsilon, T_\psi) \to E_{\bar{N}, \tilde{\bar{N}}}(\bar{R}, \bar{T}) \tag{5.7.1}$$

assigning to $(R_\varepsilon, x, T_\psi) \in E_N(R_\varepsilon, T_\psi)$ the $(\bar{N}, \tilde{\bar{N}})$-exomorphism $(\bar{R}, \bar{x}, \bar{T})_{\tilde{\bar{N}}}^{\tilde{}}$ (cf. 2.8). Recall that we denote respectively by $C_N^p(R_\varepsilon)$ and $C_{\bar{N}}^p(\bar{R}, \tilde{\bar{N}})$ the inverse images of

[6] Notice that, in Lemma 1.12 of [1], the hypothesis on P is irrelevant

$O_p(E_N(R_\varepsilon))$ in $N_N(R_\varepsilon)$ and $O_p(E_{\bar{N},\tilde{\bar{N}}}(\bar{R}))$ in $N_{\bar{N}}(\bar{R})$ (cf. 2.8), and that there is a unique point μ of $C_N^p(R_\varepsilon)$ on B such that $R_\varepsilon \subset C_N^p(R_\varepsilon)_\mu$ (cf. 2.16.2).

Theorem 5.8. *With the hypothesis and notation above:*

5.8.1. We have $\overline{O^p(C_N(R_\varepsilon))} = O^p(C_{\bar{N}}(\bar{R}, \tilde{\bar{N}}))$ *and the maps 5.7.1 are surjective.*

5.8.2. The pointed group R_ε *is a defect pointed group of* $C_N^p(R_\varepsilon)_\mu$ *if and only if the p-group* $\bar{R}$ *is a Sylow p-subgroup of* $C_{\bar{N}}^p(\bar{R}, \tilde{\bar{N}})$, *and then the maps 5.7.1 are bijective.*

Remark 5.9. Notice that, without any hypothesis on b, the unique point λ of $Q \cdot C_H(Q)$ on B such that $Q_\delta \subset Q \cdot C_H(Q)_\lambda$ (cf. 2.16.2) determines a block of $Q \cdot C_H(Q)$ with *nilpotent local structure* (cf. [18], 1.7) and therefore Theorem 5.8 always applies to the group $N_G(Q_\delta)$ and the normal subgroup $Q \cdot C_H(Q)$.

Proof. As $\overline{C_N(R_\varepsilon)} \subset C_{\bar{N}}(\bar{R}, \bar{N})$, we have already $\overline{O^p(C_N(R_\varepsilon))} \subset O^p(C_{\bar{N}}(\bar{R}, \tilde{\bar{N}}))$. If $\bar{x} \in \bar{N}$ is such that $\bar{R}^{\bar{x}} \subset \bar{T}$ we have $(R \cdot H)^x \subset T \cdot H$ where $x \in N_G(Q_\delta)$ lifts $\bar{x}$; but β is still a point of both $R \cdot H$ and $T \cdot H$ on B, and therefore $(R \cdot H_\beta)^x \subset T \cdot H_\beta$; moreover, it follows from Proposition 5.3 applied to both $R \cdot H$ and $T \cdot H$ (cf. Remark 5.4) that R_ε and T_ψ are respectively defect pointed groups of $R \cdot H_\beta$ and $T \cdot H_\beta$; hence we may choose the element x fulfilling $(R_\varepsilon)^x \subset T_\psi$, which proves that the maps 5.7.1 are surjective. Moreover, if $\bar{x}$ is a p'-element of $C_{\bar{N}}(\bar{R}, \tilde{\bar{N}})$, we may assume that x is also a p'-element which normalizes R_ε (arguing as above and eventually replacing x by a suitable power of x); then, as $\bar{x} \in \mathrm{Ker}(\bar{\eta})$ (cf. 5.6.2), x centralizes Q and, since $[x, R] \subset H \cap R = Q$ (cf. prop. 5.3), it follows that x centralizes R (cf. [9], K. I, 4.4), whence $x \in O^p(C_N(R_\varepsilon))$.

In particular, we have actually proved that $N_N(R_\varepsilon)$ maps onto $N_{\bar{N}}(\bar{R})$ and that the kernel of the induced group homomorphism

$$E_N(R_\varepsilon) \to E_{\bar{N},\tilde{\bar{N}}}(\bar{R})$$

is a p-group, whence we get $\overline{C_N^p(R_\varepsilon)} = C_{\bar{N}}^p(\bar{R}, \tilde{\bar{N}})$; hence, it follows from 5.5.2 that $\bar{R}$ is a Sylow p-subgroup of $C_{\bar{N}}^p(\bar{R}, \tilde{\bar{N}})$ if and only if R_ε is a defect pointed group of $C_N^p(R_\varepsilon)_\mu$. In that case, if $(R_\varepsilon, x, T_\psi)$ and $(R_\varepsilon, y, T_\psi)$ are two N-exomorphisms such that $(\bar{R}, \bar{x}, \bar{T})_{\tilde{\bar{N}}} = (\bar{R}, \bar{y}, \bar{T})_{\tilde{\bar{N}}}$, we have $\bar{y} = \overline{zxu}$ where $u \in T$ and $z \in R \cdot C_N(R_\varepsilon)$ since $\bar{R} \cdot C_{\bar{N}}(\bar{R}, \tilde{\bar{N}}) = C_{\bar{N}}^p(\bar{R}, \tilde{\bar{N}}) = \overline{C_N^p(R_\varepsilon)} = \bar{R} \cdot \overline{C_N(R_\varepsilon)}$ (cf. 5.5.2); in particular, we get $R^y = (R \cdot H)^y \cap T = (R \cdot H)^{xu} \cap T = R^{xu}$ since $T \cap H = Q$ (cf. prop. 5.3); hence, as T_ψ contains both $(R_\varepsilon)^y$ and $(R_\varepsilon)^{xu}$, the difference $yu^{-1}x^{-1}$ belongs to $N_G(R_\varepsilon) \cap R \cdot C_N(R_\varepsilon) \cdot H$ (cf. Prop. 5.3), and therefore we have $(R_\varepsilon, x, T_\psi) = (R_\varepsilon, y, T_\psi)$ by 5.6.1.

5.10. If $x \in N_G(Q_\delta)$ we know by Proposition 5.5 that there is a unique $\psi \in \mathscr{LP}_B(P \cap P^x)$ such that $(P \cap P^x)_\psi \subset P_\gamma$ and we denote the local pointed group $(P \cap P^x)_\psi$ by $P_\gamma \cap (P_\gamma)^x$. Although this notation is *not necessarily* symmetric, it becomes symmetric if x normalizes $P_\gamma \cap (P_\gamma)^x$ and then this local pointed group is the biggest containing Q_δ and contained in both P_γ and $(P_\gamma)^x$. In the sequel we employ this notation *only* in the symmetric case. Notice that the family $\{(P_\gamma \cap (P_\gamma)^x, 1, x, P_\gamma)\}$ where x runs over the set of elements of $N_G(Q_\delta)$ such that

$P_\gamma \cap (P_\gamma)^x = (P_\gamma \cap (P_\gamma)^x)^x$ is a *generator set* of the N_β-biexomorphisms (cf. 2.16.1, 2.17.1, 2.18.4 and Lemma 5.12).

Corollary 5.11. *With the hypothesis and notation above, let X be a subset of $N_G(Q_\delta)$ such that $P_\gamma \cap (P_\gamma)^x = (P_\gamma \cap (P_\gamma)^x)^x$ for any $x \in X$.*

5.11.1. If $x \in X$ then $(P_\gamma \cap (P_\gamma)^x, 1, x, P_\gamma)$ is an irreducible N_β-biexomorphism if and only if $(\overline{P \cap P^x}, 1, \bar{x}, \bar{P})^{\tilde{}}_{\bar{N}}$ is an irreducible $(\bar{N}, \tilde{\bar{N}})$-biexomorphism, and then we have $\overline{P \cap P^x} = \bar{P} \cap \bar{P}^{\bar{x}}$.

5.11.2. The family $\mathfrak{C} = \{(P_\gamma \cap (P_\gamma)^x, 1, x, P_\gamma)\}_{x \in X}$ is a generator set of the N_β-biexomorphisms if and only if the family $\bar{\mathfrak{C}} = \{(\overline{P \cap P^x}, 1, \bar{x}, \bar{P})^{\tilde{}}_{\bar{N}}\}_{\bar{x} \in \bar{X}}$ is a generator set of the $(\bar{N}, \tilde{\bar{N}})$-biexomorphisms.

Proof. Let $x \in X$ and assume that $R_\varepsilon = P_\gamma \cap (P_\gamma)^x$; it follows from 2.17.2, 2.18.4 and 5.8.2 that to prove 5.11.1 we may assume that R_ε is a defect pointed group of $C_N^p(R_\varepsilon)_\mu$ (cf. 5.7); then, by 5.8.2 we have $E_N(R_\varepsilon) \cong E_{\bar{N}, \tilde{\bar{N}}}(\bar{R})$ and statement 5.11.1 follows from 2.18.4.

On the other hand, it follows from 2.16.1 and 5.11.1 that to prove 5.11.2 we may assume that all the elements of $\mathfrak{C}$ and $\bar{\mathfrak{C}}$ are irreducible, and in particular that $\overline{P \cap P^x} = \bar{P} \cap \bar{P}^{\bar{x}}$ for any $x \in X$, since

$$(\overline{P \cap P^x}, 1, \bar{x}, \bar{P})^{\tilde{}}_{\bar{N}} = (\overline{P \cap P^x}, 1, \bar{P} \cap \bar{P}^{\bar{x}^{-1}})^{\tilde{}}_{\bar{N}} (\bar{P} \cap \bar{P}^{\bar{x}^{-1}}, 1, \bar{x}, \bar{P})^{\tilde{}}_{\bar{N}} .$$

As $\bar{P}$ is a Sylow p-subgroup of $\bar{N}$ (cf. Prop. 5.3) and the maps 5.7.1 are surjective (cf. 5.8.1), it is quite clear that if $\mathfrak{C}$ is a generator set of the N_β-biexomorphisms then $\bar{\mathfrak{C}}$ is a generator set of the $(\bar{N}, \tilde{\bar{N}})$-biexomorphisms. Conversely, assume that $\bar{\mathfrak{C}}$ is a generator set; by 2.17.2, 2.18.4 and Lemma 5.12 below, it suffices to show that an N_β-biexomorphism $(R_\varepsilon, 1, x, P_\gamma)$ is a linear combination of $\mathfrak{C}$ (cf. 2.15.3) assuming that R_ε is a defect pointed group of $C_N^p(R_\varepsilon)_\mu$ (cf. 5.7); but, by hypothesis, the $(\bar{N}, \tilde{\bar{N}})$-biexomorphism $(\bar{R}, 1, \bar{x}, \bar{P})^{\tilde{}}_{\bar{N}}$ is a linear combination of $\bar{\mathfrak{C}}$, and therefore there are ordered families $\{\bar{\xi}_i\}_{i \in I}$ and $\{\bar{\zeta}_i\}_{i \in I}$ of $(\bar{N}, \tilde{\bar{N}})$-exomorphisms and $\{\bar{x}_i\}_{i \in I}$ of elements of $\bar{X}$ such that (cf. 2.15.3)

$$(\bar{R}, 1, \bar{x}, \bar{P})^{\tilde{}}_{\bar{N}} = \sum_{i \in I} \bar{\xi}_i (\bar{P} \cap \bar{P}^{\bar{x}_i}, 1, \bar{x}_i, \bar{P})^{\tilde{}}_{\bar{N}} \bar{\zeta}_i ;$$

now, choosing liftings $x_i \in X$ of $\bar{x}_i$, $\xi_i \in E_N(R_\varepsilon, P_\gamma \cap (P_\gamma)^{x_i})$ of $\bar{\xi}_i$, and $\zeta_i \in E_N(P_\gamma, P_\gamma)$ of $\bar{\zeta}_i$ (cf. 5.8.1), it follows from 5.8.2 that we have

$$(R_\varepsilon, 1, x, P_\gamma) = \sum_{i \in I} \xi_i (P_\gamma \cap (P_\gamma)^{x_i}, 1, x_i, P_\gamma) \zeta_i .$$

Lemma 5.12. *With the hypothesis and notation above, let T_ψ be a local pointed group on B such that $T_\psi \subset P_\gamma$ and ν the point of $C_N^p(T_\psi)$ on B such that $T_\psi \subset C_N^p(T_\psi)_\nu$. If T_ψ is a defect pointed group of $C_N^p(T_\psi)_\nu$ then $Q_\delta \subset T_\psi$.*

Proof. It suffices to prove that $N_H(T_\psi)/C_H(T)$ is a p-group; indeed, as $N_H(T_\psi)$ is normal in $N_N(T_\psi)$, in that case we have $N_H(T_\psi) \subset C_N^p(T_\psi)$ and *a fortiori*

$N_{T\cdot H}(T_\psi) \subset C_N^p(T_\psi)$; thus, if λ is the point of $N_{T\cdot H}(T_\psi)$ on B such that $T_\psi \subset N_{T\cdot H}(T_\psi)_\lambda \subset C_N^p(T_\psi)_\nu$ (cf. 2.16.2), then T_ψ is still a defect pointed group of $N_{T\cdot H}(T_\psi)_\lambda$; but, since $T_\psi \subset P_\gamma \subset N_\beta$, we have $b\psi = \psi$ and so $b\lambda = \lambda$, whence $N_{T\cdot H}(T_\psi)_\lambda \subset T\cdot H_\beta$; consequently, T_ψ is also a defect pointed group of $T\cdot H_\beta$ (cf. [15], Cor. 1.4), whence $Q_\delta \subset T_\psi$ by Propositions 5.3 and 5.5.

Set $U = T \cap H$; as H is a P-stable $\mathcal{O}$-basis of B, there is $\varphi \in \mathscr{L}\mathscr{P}_B(U)$ such that $U_\varphi \subset T_\psi$ (cf. [1], Prop. 1.5); thus, as $U_\varphi \subset T_\psi \subset P_\gamma \subset N_\beta$, we have $U_\varphi \subset H_\beta$ (since $b\varphi = \varphi$); therefore, $E_H(U_\varphi)$ is a p-group (cf. [18], 1.7.2) and the block of $C_H(U)$ associated with φ is still nilpotent (cf. [2], Th. 1.2), which implies that $N_H(T_\psi) \subset N_H(U_\varphi)$ (cf. [1], Th. 1.8 and [2], Th. 1.2). So, for any p'-element s of $N_H(T_\psi)$, we have $[s, T] \subset H \cap T = U$ and $s \in C_H(U)$, whence s centralizes T (cf. [9], K. I, 4.4).

6. On p-extensions of nilpotent blocks

6.1. Let G be a finite group and H a normal subgroup of G such that G/H is a p-group. In this section we will study the unique block of G lying over a *nilpotent block* of H (cf. [18], 1.7): the main tool to carry out our purpose are the *semicovering exomorphisms* between G-algebras introduced in Sect. 3. Precisely, set $A = \mathcal{O}G$ and $B = \mathcal{O}H$, and consider first A and B just as G-algebras: the key point is that the inclusion $B \subset A$ determines a *strict semicovering exomorphism*, as we show below. Actually, this fact is useful to relate *any* block b of H with the unique block of G lying over b.

Proposition 6.2. *The G-algebra exomorphism determined by the inclusion $B \subset A$ is a strict semicovering. In particular, for any pointed p-group P_{γ° on A there is a non empty set $\mathscr{S}(\gamma^\circ) \subset \mathscr{P}_B(P)$ such that for any $\gamma \in \mathscr{S}(\gamma^\circ)$ we have $\gamma \subset \gamma^\circ$, γ is local if and only if γ° is local, and then $\mathscr{S}(\gamma^\circ) = \{\gamma^z\}_{z \in C_G(P)}$. Moreover, we have*

$$\mathscr{P}_B(P) = \bigcup_{\gamma^\circ \in \mathscr{P}_A(P)} \mathscr{S}(\gamma^\circ) \tag{6.2.1}$$

Proof. By Theorem 3.16, to prove the first statement it suffices to show that, for any p-subgroup P of G, the $\mathcal{k}$-algebra exomorphism $B(P) \to A(P)$ induced by the inclusion $B \subset A$ is a *strict semicovering*; but $A(P) \cong \mathcal{k}C_G(P)$ and $B(P) \cong \mathcal{k}C_H(P)$ (cf. [19], 2.9.2), and the $\mathcal{k}$-algebra exomorphism above is determined by these isomorphisms and by the inclusion $C_H(P) \subset C_G(P)$; so, it suffices to apply Example 3.9. Moreover, the second statement follows from 3.10, 3.3.2 and 3.15.3, and the equality 6.2.1 follows from 3.10 and the last statement of Proposition 3.3. Finally, the transitivity of $C_G(P)$ follows from 3.10.2 since $A(P_{\gamma^\circ})^{C_G(P)} \cong \mathcal{k}$ (cf. [19], 2.9.2).

The following consequence is more or less well-known.

Corollary 6.3. *We have $\mathscr{P}_A(G) = \mathscr{P}_B(G)$. Moreover, let α be a point of G on A, P_γ a pointed p-group on B and γ° the point of P on A such that $\gamma \subset \gamma^\circ$; then P_γ is a defect pointed group of G_α on B if and only if P_{γ° is a defect pointed group of G_α on A.*

Proof. By Proposition 6.2, any idempotent of A has an $A^\times$-conjugate contained in B (cf. 3.10 and 3.2); in particular, as $A^G = Z(A)$, any idempotent of A^G belongs to B^G and therefore $\mathscr{P}_A(G) = \mathscr{P}_B(G)$. Moreover, if P_γ is a defect pointed group of G_α on B then $\alpha \subset \mathrm{Tr}_P^G(B^P \cdot \gamma \cdot B^P)$ and *a fortiori* $\alpha \subset \mathrm{Tr}_P^G(A^P \cdot \gamma^\circ \cdot A^P)$ which implies that P_{γ° is a defect pointed group of G_α on A (cf. 3.15.3 and [15], Th. 1.2). Conversely, if P_{γ° is a defect pointed group of G_α on A then P_{γ° is a maximal local pointed group on A, and therefore P_γ is a maximal local pointed group on B (cf. 3.15.3) such that $P_\gamma \subset G_\alpha$.

6.4. From now on, we consider A endowed with B as a *G-algebra extension* (cf. 2.9). Let α and β be respectively points of G and H on B such that $H_\beta \subset G_\alpha$; by Corollary 6.3, α is also a point of G on A and the corresponding block of G lies over the block of H determined by β. Moreover, set $N = N_G(H_\beta)$ and let P_γ and Q_δ be respectively defect pointed groups of G_α and H_β such that $Q_\delta \subset P_\gamma \subset N_\beta$ (cf. 5.2); by Proposition 5.3 we have

$$Q = P \cap H \qquad \text{and} \qquad N = P \cdot H; \tag{6.4.1}$$

it follows from Corollary 6.3 that the *interior* G-algebra underlying the *G-algebra extension* A_α (cf. 2.11) is an embedded algebra associated with G_α on A (cf. [19], 2.13), and that the *interior* P-algebra underlying the *P-algebra extension* A_γ (cf. 2.11) is a *source* of the *interior* G-algebra underlying A_α (cf. [15], Def. 3.2).

The next result has already been proved – in an unessentially weaker form – in [3].

Proposition 6.5. *With the notation above, α determines a nilpotent block of G if and only if β determines a nilpotent block of H.*

Proof. If α determines a nilpotent block of G and R_ε is a local pointed group on B such that $R_\varepsilon \subset H_\beta$, there is a unique $\varepsilon^0 \in \mathscr{L}\mathscr{P}_A(R)$ such that $\varepsilon \subset \varepsilon^0$ (cf. Prop. 6.2) and we have $R_{\varepsilon^0} \subset G_\alpha$; hence, $E_G(R_{\varepsilon^0})$ is a p-group (cf. [18], 1.7.2) and $N_H(R_\varepsilon) \subset N_G(R_{\varepsilon^0})$, which implies that $E_H(R_\varepsilon)$ is still a p-group. Conversely, assume that β determines a nilpotent block of H; if R_ε is a local pointed group on B such that $R_\varepsilon \subset P_\gamma$ and if $\varepsilon^0 \in \mathscr{L}\mathscr{P}_A(R)$ contains ε (cf. Prop. 6.2), it suffices to prove that $E_G(R_{\varepsilon^0})$ is a p-group (cf. Prop. 6.2, Cor. 6.3 and [18], Th. 3.8). It follows from 6.4.1 and 5.11.2 that the family $\{(P_\gamma, 1, u, P_\gamma)\}_{u \in P}$ is a generator set of the N_β-biexomorphisms, and therefore it is easily checked that $E_N(R_\varepsilon, P_\gamma) = E_P(R_\varepsilon, P_\gamma)$ (cf. 2.15.3); hence, as $N_G(R_{\varepsilon^0}) = C_G(R) \cdot N_G(R_\varepsilon)$ (cf. Prop. 6.2), we get $N_G(R_{\varepsilon^0}) = C_G(R) \cdot N_P(R_\varepsilon)$ (recall that $E_G(R_\varepsilon) = E_N(R_\varepsilon)$ by 5.2).

The next result restates Theorem 1.12 in the present situation. Recall that, if S is a P-algebra, the *interior* P-algebra SP is the free S-module over P endowed with the evident multiplication and the canonical map from P (cf. 2.9).

Theorem 6.6. *With the notation above, assume that β determines a nilpotent block of H. There is an $\mathcal{O}$-simple P-algebra S such that, considering SP endowed with SQ as a P-algebra extension, we have a unique P-algebra extension exoisomorphism*

$$A_\gamma \cong SP\,. \tag{6.6.1}$$

Moreover, the P-algebra S is unique up to isomorphism and has a P-stable $\mathcal{O}$-basis which contains the unity as the unique P-fixed element.

Proof. By Proposition 6.5, α determines a nilpotent block of G, and therefore as the *interior* P-algebra underlying A_γ is a *source algebra* of the block (cf. 6.4), it follows from the main theorem of [18] that there is an $\mathcal{O}$-simple P-algebra S, unique up to isomorphism, such that $A_\gamma \cong SP$ as *interior* P-algebras and fulfilling all the conditions above. Moreover, we know (cf. [18], 1.8.1) that there is a unique group homomorphism $\rho: P \to S^\times$ lifting the action of P on S such that $\det(\rho(u)) = 1$ for any $u \in P$, and that $SP \cong S \otimes_{\mathcal{O}} \mathcal{O}P$ considering S endowed with ρ as an *interior* P-algebra; hence, since we have a (unique) embedding from the trivial *interior* P-algebra $\mathcal{O}$ to $S \otimes_{\mathcal{O}} S^0$ (cf. [18], 5.7), we get an interior P-algebra embedding

$$\mathcal{O}P \to S^0 \underset{\mathcal{O}}{\otimes} SP \tag{6.6.1}$$

which is an embedded algebra (cf. 2.11 or [18], 2.10) associated with the unique local point of P on $S^0 \otimes_{\mathcal{O}} SP$ (cf. [18], 5.3.1).

On the other hand, by Propositions 6.2 and 3.14 the inclusion $B_\gamma \subset A_\gamma$ determines a strict semicovering exomorphism of P-algebras. Then we claim that

6.6.2. the inclusions $S^0 \otimes_{\mathcal{O}} B_\gamma \subset S^0 \otimes_{\mathcal{O}} A_\gamma$ and $S \otimes_{\mathcal{O}} S^0 \otimes_{\mathcal{O}} B_\gamma \subset S \otimes_{\mathcal{O}} S^0 \otimes_{\mathcal{O}} A_\gamma$ determine strict semicovering exomorphisms of P-algebras.

Since B is a direct summand of A as $\mathcal{O}P$-modules, for any subgroup R of P we get canonical inclusions

$$(S^0 \underset{\mathcal{O}}{\otimes} B_\gamma)(R) \subset (S^0 \underset{\mathcal{O}}{\otimes} A_\gamma)(R) \qquad \text{and} \qquad (S \underset{\mathcal{O}}{\otimes} S^0 \underset{\mathcal{O}}{\otimes} B_\gamma)(R) \subset (S \underset{\mathcal{O}}{\otimes} S^0 \underset{\mathcal{O}}{\otimes} A_\gamma)(R)$$

and, by Theorem 3.16, it suffices to prove that these inclusions determine *strict semicovering exomorphisms* of k-algebras; but the inclusion $B_\gamma(R) \subset A_\gamma(R)$ does it (cf. Th. 3.16); hence, by Corollary 5.8 in [18], the following inclusions do it too

$$S(R)^0 \underset{k}{\otimes} B_\gamma(R) \subset S(R)^0 \underset{k}{\otimes} A_\gamma(R) \qquad \text{and}$$

$$(S \underset{\mathcal{O}}{\otimes} S^0)(R) \underset{k}{\otimes} B_\gamma(R) \subset (S \underset{\mathcal{O}}{\otimes} S^0)(R) \underset{k}{\otimes} A_\gamma(R)$$

and now 6.6.2 follows from Proposition 5.6 in [18] (which is still true for P-algebras).

Moreover, by 5.3.1 in [18] (which is also true for P-algebras) we know that P has unique local points $S^0 \times \gamma$ on $S^0 \otimes_{\mathcal{O}} B_\gamma$ and $S \otimes_{\mathcal{O}} S^0 \times \gamma$ on $S \otimes_{\mathcal{O}} S^0 \otimes_{\mathcal{O}} B_\gamma$ (where we still denote by γ the unique point of P on B_γ). So, by 6.6.2, it follows from Proposition 3.3 that $S^0 \times \gamma$ and $S \otimes_{\mathcal{O}} S^0 \times \gamma$ are respectively contained in the unique local points of P on $S^0 \otimes_{\mathcal{O}} A_\gamma$ and $S \otimes_{\mathcal{O}} S^0 \otimes_{\mathcal{O}} A_\gamma$ (cf. 3.15.3 and [18], 5.3.1). In particular, since the *interior* P-algebra underlying the embedded algebra extension $(S^0 \otimes_{\mathcal{O}} A_\gamma)_{S^0 \times \gamma} \to S^0 \otimes_{\mathcal{O}} A_\gamma$ associated with $P_{S^0 \times \gamma}$ (cf. 2.11) is an embedded algebra associated with the unique local point of P on $S^0 \otimes_{\mathcal{O}} A_\gamma$, and since $S^0 \otimes_{\mathcal{O}} A_\gamma \cong S^0 \otimes_{\mathcal{O}} S \otimes_{\mathcal{O}} \mathcal{O}P$ as *interior* P-algebras, it follows from 6.6.1 and the

uniqueness of embedded algebras (cf. 2.11.1 or [18], 2.10.1) that

$$(S^0 \underset{\mathcal{O}}{\otimes} A)_{S^0 \times \gamma} \cong \mathcal{O}P \tag{6.6.3}$$

as *interior* P-algebras.

But such an isomorphism maps necessarily $(S^0 \otimes_{\mathcal{O}} B_\gamma)_{S^0 \times \gamma}$ onto $\mathcal{O}Q$; indeed, it is quite clear that $P \cap B = Q$ and $A = \oplus_{u \in T} uB$, where T is a left transversal to Q in P, which implies successively that $Q \cdot B_\gamma = B_\gamma$ and $A = \oplus_{u \in T} u \cdot B_\gamma$, that

$$Q \cdot (S^0 \underset{\mathcal{O}}{\otimes} B_\gamma) = S^0 \underset{\mathcal{O}}{\otimes} B_\gamma \qquad \text{and} \qquad S^0 \underset{\mathcal{O}}{\otimes} A_\gamma = \bigoplus_{u \in T} u \cdot (S^0 \underset{\mathcal{O}}{\otimes} B_\gamma),$$

and finally that

$$Q \cdot (S^0 \underset{\mathcal{O}}{\otimes} B_\gamma)_{S^0 \times \gamma} = (S^0 \underset{\mathcal{O}}{\otimes} B_\gamma)_{S^0 \times \gamma} \quad \text{and} \quad (S^0 \underset{\mathcal{O}}{\otimes} A_\gamma)_{S^0 \times \gamma} = \bigoplus_{u \in T} u \cdot (S^0 \underset{\mathcal{O}}{\otimes} B_\gamma)_{S^0 \times \gamma};$$

so, the image C of $(S^0 \otimes_{\mathcal{O}} B_\gamma)_{S^0 \times \gamma}$ in $\mathcal{O}P$ through the bijective $\mathcal{O}$-linear map 6.6.3 fulfils $Q \cdot C = C$ and $\mathcal{O}P = \oplus_{u \in T} u \cdot C$ which forces $C = \mathcal{O}Q$. Hence, considering $\mathcal{O}P$ endowed with $\mathcal{O}Q$ as a *P-algebra extension*, the map 6.6.3 is an isomorphism of *P-algebra extensions*, and therefore we have

$$S \underset{\mathcal{O}}{\otimes} (S^0 \underset{\mathcal{O}}{\otimes} A_\gamma)_{S^0 \times \gamma} \cong SP \tag{6.6.4}$$

as *P-algebra extensions*. On the other hand, the embeddings $\mathcal{O} \to S \otimes_{\mathcal{O}} S^0$ (as *interior* P-algebras) and $(S^0 \otimes_{\mathcal{O}} A_\gamma)_{S^0 \times \gamma} \to S^0 \otimes_{\mathcal{O}} A_\gamma$ (as *P-algebra extensions*) induce the P-algebra extension embeddings

$$S \underset{\mathcal{O}}{\otimes} (S^0 \underset{\mathcal{O}}{\otimes} A_\gamma)_{S^0 \times \gamma} \to S \underset{\mathcal{O}}{\otimes} S^0 \underset{\mathcal{O}}{\otimes} A_\gamma \leftarrow A_\gamma \tag{6.6.5}$$

which are two embedded extension algebras associated with the unique local point $S \otimes_{\mathcal{O}} S^0 \times \gamma$ of P on $S \otimes_{\mathcal{O}} S^0 \otimes_{\mathcal{O}} B_\gamma$ (cf. 2.11). Consequently, by 6.6.4, 6.6.5 and 2.11.1 we have $A_\gamma \cong SP$ as *P-algebra extensions*.

Now it suffices to prove that any P-algebra extension automorphism f of SP is induced by an element of $((SQ)^P)^\times$. Set

$$W = 1 + \sum_{v \in Q} \mathrm{Ker}(\mathrm{tr})(1 - \rho(v)^{-1} v)$$

where $\mathrm{tr}: S \to \mathcal{O}$ denotes the trace map; by Proposition 6.8 of [18] we have $W \subset (SQ)^\times$ and there is a unique $w \in W$ such that $f(S) = S^w$; thus P fixes w (since P normalizes W and $f(S)$). So we may assume that $f(S) = S$, and then there is $s \in S^\times$ such that $f(t) = t^s$ for any $t \in S$; in particular, the uniqueness of ρ (cf. [18], 1.8.1) forces $f \circ \rho = \rho$ and therefore P fixes s. Hence, we may assume that $f(t) = t$ for any $t \in S$, and then $f = \mathrm{id}_{SP}$.

7. The projective equivalence between nilpotent block extensions

7.1. Let G be a finite group, H a normal subgroup of G and b a nilpotent block of H (cf. [18], 1.7). As we say in the introduction, our main purpose is to study the blocks of G lying over b. Clearly we have such a situation whenever H is a p-group and $b = 1$, and roughly speaking, our main results (cf. Th. 1.8 and Th. 1.12) state

that we can always reduce the general situation to a central p'-extension of such a particular case. In this section we give some characterizations of this reduction. But rather than to discuss just the reduction procedure, we will include it in a full equivalence relation which will allow us to invoke symmetry in all the arguments. This relation will be named *projective equivalence* to emphasize that central p'-extensions are neglected.

7.2. Take again the notation introduced in 5.1 and 5.6. Similarly, let G' be a finite group, H' a normal subgroup of G', A' the G'-*algebra extension* formed by $\mathcal{O}G'$ endowed with $B' = \mathcal{O}H'$, b' a nilpotent block of H', β' the point $\{b'\}$ of H' on B', α' the unique point of G' on B' such that $H'_{\beta'} \subset G'_{\alpha'}$, N' the normalizer of $H'_{\beta'}$ in G', $P'_{\gamma'}$ and $Q'_{\delta'}$ respective defect pointed groups of $G'_{\alpha'}$ and $H'_{\beta'}$ fulfilling $Q'_{\delta'} \subset P'_{\gamma'} \subset N'_{\beta'}$, and $\bar{N}'$ the quotient N'/H'. Recall that $\bar{N}' \cong N_{G'}(Q'_{\delta'})/Q' \cdot C_{H'}(Q')$ (cf. 1.7.2) and that $\bar{X}'$ denotes the image of $X' \subset N'$ in $\bar{N}'$. As in 1.6, we consider respectively N and N', endowed with the actions on the sets $\bar{N}$ and $\bar{N}'$ induced by left multiplication, as $\bar{N}$- and $\bar{N}'$-*permutation groups* (cf. 2.4).

7.3. Let $\bar{\sigma}: \bar{N} \cong \bar{N}'$ and $\tau: P \cong P'$ be group isomorphisms. We say that the G- and G'-*algebra extensions* A_α and $A'_{\alpha'}$ are *projectively equivalent through* $\bar{\sigma}$ *and* τ if the following conditions hold:

7.3.1. We have $\bar{\sigma}(\bar{u}) = \overline{\tau(u)}$ *for any* $u \in P$.

7.3.2. For any local pointed groups R_ε *on* B *and* $R'_{\varepsilon'}$ *on* B' *such that* $Q_\delta \subset R_\varepsilon \subset P_\gamma$, $Q'_{\delta'} \subset R'_{\varepsilon'} \subset P'_{\gamma'}$ *and* $\tau(R) = R'$, *the bijection from* $\tilde{\mathrm{H}}\mathrm{om}_{\bar{N}}(R, P)$ *to* $\tilde{\mathrm{H}}\mathrm{om}_{\bar{N}'}(R', P')$ *induced by* $\bar{\sigma}$ *and* τ *maps* $E_{N,\bar{N}}(R_\varepsilon, P_\gamma)$ *onto* $E_{N',\bar{N}'}(R'_{\varepsilon'}, P'_{\gamma'})$.

Then $A'_{\alpha'}$ and A_α are projectively equivalent through $\bar{\sigma}^{-1}$ and τ^{-1}. We say that A_α and $A'_{\alpha'}$ are *projectively equivalent* if there are group isomorphisms $\bar{N} \cong \bar{N}'$ and $P \cong P'$ fulfilling conditions 7.3.1 and 7.3.2. This relation is clearly symmetric and transitive, and does not depend on the choice of the defect groups. Moreover, condition 7.3.2 implies that:

7.3.3. We have $\bar{\sigma}(\overline{C_N(R_\varepsilon)}) = \overline{C_{N'}(R'_{\varepsilon'})}$.

Indeed, by 7.3.2, for any $z \in C_N(R_\varepsilon)$ there is $x' \in N'$ such that $(R'_{\varepsilon'})^{x'} \subset P'_{\gamma'}$, $\tau(u)^{x'} = \tau(u)$ for any $u \in P$ and $\bar{x}'^{-1}\bar{\sigma}(\bar{n}) = \bar{\sigma}(\bar{z}^{-1}\bar{n})$ for any $\bar{n} \in \bar{N}$ (cf. 2.5 and 2.8); therefore we have $x' \in C_{N'}(R'_{\varepsilon'})$ (cf. Prop. 5.5) and $\bar{x}' = \bar{\sigma}(\bar{z})$, whence $\bar{\sigma}(\bar{z}) \in \overline{C_{N'}(R'_{\varepsilon'})}$; now we get 7.3.3 by symmetry.

7.4. First of all, we will describe the projective equivalence between A_α and $A'_{\alpha'}$ in terms of the full subcategories $\mathfrak{L}$ and $\mathfrak{L}'$ of the *local categories* of B_β and $B'_{\beta'}$ (cf. 2.8) formed respectively by the local pointed groups on B and B' which contain defect pointed groups of H_β and $H'_{\beta'}$. Consider the evident functors

$$\mathfrak{f}: \mathfrak{L} \to \mathfrak{N} \qquad \text{and} \qquad \mathfrak{f}': \mathfrak{L}' \to \mathfrak{N}'$$

from $\mathfrak{L}$ and $\mathfrak{L}'$ to the categories $\mathfrak{N}$ and $\mathfrak{N}'$ formed respectively by the $\bar{N}$- and $\bar{N}'$-*permutation groups* and the *permutation group exomorphisms* (cf. 2.5). Notice that any bijection from $\bar{N}$ onto $\bar{N}'$ induces an equivalence of categories from $\mathfrak{N}$ to $\mathfrak{N}'$.

Proposition 7.5. *With the notation above, A_α and $A'_{\alpha'}$ are projectively equivalent if and only if there are a group isomorphism $\bar\sigma\colon \bar N \cong \bar N'$, an equivalence of categories $\mathfrak{t}\colon \mathfrak{Q} \sim \mathfrak{Q}'$ and a natural isomorphism $\tilde\tau\colon \mathfrak{s}\circ\mathfrak{f} \cong \mathfrak{f}'\circ\mathfrak{t}$, where $\mathfrak{s}$ is the equivalence from $\mathfrak{R}$ to $\mathfrak{R}'$ induced by $\bar\sigma$, such that*

7.5.1 *for any object R_ε of $\mathfrak{Q}$, the $\bar N'$-permutation group exomorphism $\tilde\tau(R_\varepsilon)$ has a representative inducing the identity on $\bar N'$.*

Proof. Assume first that there are $\bar\sigma$, $\mathfrak{t}$ and $\tilde\tau$ as above fulfilling condition 7.5.1. Since $\mathfrak{t}(P_\gamma)$ is a maximal object of $\mathfrak{Q}'$, we may assume that $\mathfrak{t}(P_\gamma) = P'_{\gamma'}$. We claim that, choosing a representative $(\tau_\gamma, \mathrm{id}_{\bar N'})$ of $\tilde\tau(P_\gamma)$ (cf. 2.5), the group isomorphisms $\bar\sigma\colon \bar N \cong \bar N'$ and $\tau_\gamma\colon P \cong P'$ fulfil conditions 7.3.1 and 7.3.2. By definition, for any $u\in P$ and any $\bar n'\in \bar N'$ we have $u\cdot\bar n' = \tau_\gamma(u)\cdot\bar n'$ (cf. 2.5), whence $\overline{\bar\sigma(\bar u)}\bar n' = \overline{\tau_\gamma(u)}\bar n'$ which proves 7.3.1; in particular, we get $\tau_\gamma(Q) = Q'$ (cf. Prop. 5.3).

Moreover, let R_ε and $R'_{\varepsilon'}$ be respectively local pointed groups on B and B' such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$, $Q'_{\delta'} \subset R'_{\varepsilon'} \subset P'_{\gamma'}$ and $\tau_\gamma(R) = R'$, set $\mathfrak{t}(R_\varepsilon) = R''_{\varepsilon''}$ and choose a representative $(\tau_\varepsilon, \mathrm{id}_{\bar N'}) =$ of $\tilde\tau(R_\varepsilon)$. For any $(N, \bar N)$-exomorphism $(R_\varepsilon, x, P_\gamma)_{\bar N}$, setting

$$\mathfrak{t}((R_\varepsilon, x, P_\gamma)_{\bar N}) = (R''_{\varepsilon''}, x'', P'_{\gamma'})_{\bar N'} \qquad \text{and} \qquad \mathfrak{t}((R_\varepsilon, 1, P_\gamma)_{\bar N}) = (R''_{\varepsilon''}, y'', P'_{\gamma'})_{\bar N'},$$

by the naturality of $\tilde\tau$ we may assume that (cf. 2.5)

$$\tau_\varepsilon(u)^{x''} = \tau_\gamma(u^x) \qquad \text{and} \qquad \tau_\varepsilon(u)^{y''} = \tau_\gamma(u)$$

for any $u\in R$, and that

$$\bar x''\bar n' = x\cdot\bar n' = \bar\sigma(\bar x)\bar n' \qquad \text{and} \qquad \bar y''\bar n' = \bar n'$$

for any $\bar n'\in \bar N'$; hence we have $(R''_{\varepsilon''})^{y''} = R'_{\varepsilon'}$ (cf. Prop. 5.5) and, setting $x' = y''^{-1}x''$, we get $(R'_{\varepsilon'})^{x'} \subset P'_{\gamma'}$, $\bar\sigma(\bar x) = \bar x'$ and $\tau_\gamma(u^x) = \tau_\gamma(u)^{x'}$ for any $u\in P$, which proves that the bijection from $\tilde{\mathrm{H}}\mathrm{om}_{\bar N}(R, P)$ to $\tilde{\mathrm{H}}\mathrm{om}_{\bar N'}(R', P')$ induced by $\bar\sigma$ and τ_γ maps $(R_\varepsilon, x, P_\gamma)_{\bar N}$ upon $(R'_{\varepsilon'}, x', P'_{\gamma'})_{\bar N'}$. So, this bijection maps $E_{N,\bar N}(R_\varepsilon, P_\gamma)$ to $E_{N',\bar N'}(R'_{\varepsilon'}, P'_{\gamma'})$, and by symmetry its inverse maps $E_{N',\bar N'}(R'_{\varepsilon'}, P'_{\gamma'})$ to $E_{N,\bar N}(R_\varepsilon, P_\gamma)$.

Conversely, assume that there are group isomorphisms $\bar\sigma\colon \bar N \cong \bar N'$, and $\tau\colon P \cong P'$ fulfilling conditions 7.3.1 and 7.3.2. For any object R_ε of $\mathfrak{Q}$ there are $h\in H$ and $x\in N$ such that $(Q_\delta)^h \subset R_\varepsilon$ and $(R_\varepsilon)^x \subset P_\gamma$; notice that $(Q_\delta)^{hx} = Q_\delta$ by Propositions 5.3. and 5.5. By condition 7.3.2 applied to Q_δ and $Q'_{\delta'}$, the bijection from $\tilde{\mathrm{H}}\mathrm{om}_{\bar N}(Q_\delta, P_\gamma)$ to $\tilde{\mathrm{H}}\mathrm{om}_{\bar N'}(Q'_{\delta'}, P'_{\gamma'})$ induced by $\bar\sigma$ and τ maps $(Q_\delta, hx, P_\gamma)_{\bar N}$ upon some element of $E_{N',\bar N'}(Q'_{\delta'}, P'_{\gamma'})$; that is, there is $x'\in N'$ such that $\bar\sigma(\bar x) = \bar x'$ and $\tau(v^{hx}) = \tau(v)^{x'}$ for any $v\in Q$. Moreover, by Proposition 5.5, there is a unique local pointed group $R'_{\varepsilon'}$ on B' such that $(R')^{x'} = \tau(R^x)$ and $Q'_{\delta'} \subset (R'_{\varepsilon'})^{x'} \subset P'_{\gamma'}$, and we denote by $\tau_\varepsilon\colon R \to R'$ the group isomorphism mapping $u\in R$ upon $\tau(u^x)^{x'^{-1}}$; notice that by condition 7.3.1 we have

$$\overline{\tau_\varepsilon(u)} = \overline{\tau(u^x)}^{\bar x'^{-1}} = \bar\sigma(\bar u^{\bar x})^{\bar x'^{-1}} = \bar\sigma(\bar u)$$

for any $u\in R$, and therefore $(\tau_\varepsilon, \bar\sigma)$ is a permutation group isomorphism from $(R, \bar N)$ onto $(R', \bar N')$ (cf. 2.5).

Now we consider the functor $\mathfrak{t}\colon \mathfrak{Q} \to \mathfrak{Q}'$ mapping R_ε upon $R'_{\varepsilon'}$ as above and any morphism $\tilde\varphi\in E_{N,\bar N}(R_\varepsilon, T_\psi)$ upon the unique $\bar N'$-permutation group exomorphism

$\mathfrak{t}(\tilde{\varphi})$ from R' to T', where $T'_{\psi'} = \mathfrak{t}(T_\psi)$, such that

$$\mathfrak{t}(\tilde{\varphi}) \circ \widetilde{(\tau_\varepsilon, \bar{\sigma})} = \widetilde{(\tau_\psi, \bar{\sigma})} \circ \tilde{\varphi} ,$$

which belongs to $E_{N', \bar{N}'}(R'_{\varepsilon'}, T'_{\psi'})$ by condition 7.3.2; it is quite clear by symmetry that $\mathfrak{t}$ is an equivalence of categories. Finally, denoting by $\mathfrak{s}: \mathfrak{N} \to \mathfrak{N}'$ the equivalence of categories induced by $\bar{\sigma}$, we have an evident natural isomorphism $\mathfrak{s} \circ \mathfrak{f} \cong \mathfrak{f}' \circ \mathfrak{t}$ mapping any object R_ε of $\mathfrak{Q}$ upon the $\bar{N}'$-permutation group exomorphism determined by $(\tau_\varepsilon, \mathrm{id}_{\bar{N}'})$.

7.6. Now we will prove that the whole condition 7.3.2 depends only on the behavior under the action of $\bar{\sigma}$ and τ of a "small set" of $(N, \bar{N})$-exomorphisms. Recall that for any $x \in N_G(Q_\delta)$ and any $x' \in N_{G'}(Q'_{\delta'})$ we denote respectively by $P_\gamma \cap (P_\gamma)^x$ and $P'_{\gamma'} \cap (P'_{\gamma'})^{x'}$ the unique local pointed groups $(P \cap P^x)_\varepsilon$ on B and $(P' \cap P'^{x'})_{\varepsilon'}$ on B' such that (cf. 5.10)

$$(P \cap P^x)_\varepsilon \subset P_\gamma \qquad \text{and} \qquad (P' \cap P'^{x'})_{\varepsilon'} \subset P'_{\gamma'} .$$

Notice that the family $\{P_\gamma \cap (P_\gamma)^x, 1, x, P_\gamma)\}$ where x runs over the set of elements of $N_G(Q_\delta)$ such that $P_\gamma \cap (P_\gamma)^x = (P_\gamma \cap (P_\gamma)^x)^x$ (and so $P_\gamma \cap (P_\gamma)^x = (P_\gamma)^x \cap P_\gamma$) and $\overline{P \cap P^x} = \bar{P} \cap \bar{P}^{\bar{x}}$ is a generator set of the N_β-biexomorphisms (cf. 2.16.1, 2.17.1, 2.18.4 and 5.11.1).

Theorem 7.7. *With the notation above, let $\bar{\sigma}: \bar{N} \cong \bar{N}'$ and $\tau: P \cong P'$ be group isomorphisms and X a subset of $N_G(Q_\delta)$ containing $N_G(P_\gamma)$ such that $P_\gamma \cap (P_\gamma)^x = (P_\gamma \cap (P_\gamma)^x)^x$ and $\overline{P \cap P^x} = \bar{P} \cap \bar{P}^{\bar{x}}$ for any $x \in X$. Assume that $\bar{\sigma}$ and τ fulfill condition 7.3.1 and that the family $\{(P_\gamma \cap (P_\gamma)^x, 1, x, P_\gamma)\}_{x \in X}$ is a generator set of the N_β-biexomorphisms. Then A_α and $A'_{\alpha'}$ are projectively equivalent through $\bar{\sigma}$ and τ if and only if the following conditions hold:*

7.7.1. We have $\bar{\sigma}(\overline{C_N(Q_\delta)}) = \overline{C_{N'}(Q'_{\delta'})}$.

7.7.2. For any $x \in X$ there is $x' \in N_{G'}(Q'_{\delta'})$ such that $\overline{P' \cap P'^{x'}} = \bar{P}' \cap \bar{P}'^{\bar{x}'}$, $P'_{\gamma'} \cap (P'_{\gamma'})^{x'} = (P'_{\gamma'} \cap (P'_{\gamma'})^{x'})^{x'}$, $\bar{\sigma}(\bar{x}) = \bar{x}'$ and $\tau(u^x) = \tau(u)^{x'}$ for any $u \in P \cap P^x$.

Proof. Conditions 7.7.1 and 7.7.2 are necessary; indeed, 7.3.3 implies 7.7.1; moreover, for any $x \in X$, setting $R_\varepsilon = P_\gamma \cap (P_\gamma)^x$ and considering the $(N, \bar{N})$-exomorphism $(R_\varepsilon, x, P_\gamma)_{\bar{N}}$, condition 7.3.2 implies that there is $x' \in N'$ such that $\sigma(x) = x'$, $\tau(u^x) = \tau(u)^{x'}$ for any $u \in R$ and $(R'_{\varepsilon'})^{x'} \subset P'_{\gamma'}$ where $R'_{\varepsilon'}$ is the local pointed group on B' such that $R'_{\varepsilon'} \subset P'_{\gamma'}$ and $R' = \tau(R)$ (cf. Prop. 5.5); in particular, we have

$$(R'_{\varepsilon'})^{x'} = R'_{\varepsilon'} \qquad \text{and} \qquad \bar{R}' = \overline{\tau(R)} = \bar{\sigma}(\bar{R}) = \bar{\sigma}(\bar{P} \cap \bar{P}^{\bar{x}}) = \bar{P}' \cap \bar{P}'^{\bar{x}'}$$

which implies

$$\overline{P' \cap P'^{x'}} = \bar{P}' \cap \bar{P}'^{\bar{x}'} \qquad \text{and} \qquad P'_{\gamma'} \cap (P'_{\gamma'})^{x'} = (P'_{\gamma'} \cap (P'_{\gamma'})^{x'})^{x'} .$$

Conversely, assume that conditions 7.7.1 and 7.7.2 hold. Let R_ε and $R'_{\varepsilon'}$ be respectively local pointed groups on B and B' such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$, $Q'_{\delta'} \subset R'_{\varepsilon'} \subset P'_{\gamma'}$ and $R' = \tau(R)$ (cf. Prop. 5.5), and μ the unique point of $R \cdot C_N(R_\varepsilon)$ on B such that $R_\varepsilon \subset R \cdot C_N(R_\varepsilon)_\mu$. We will prove first that:

7.7.3. If P_γ contains a defect pointed group of $R \cdot C_N(R_\varepsilon)_\mu$ we have

$$\bar{\sigma}(\overline{C_N(R_\varepsilon)}) \subset \overline{C_{N'}(R'_{\varepsilon'})} .$$

If T_ψ is a defect pointed group of $R \cdot C_N(R_\varepsilon)_\mu$, it follows from 5.5.2 that $\bar{T}$ is a Sylow p-subgroup of $\overline{R \cdot C_N(R_\varepsilon)}$, whence $\bar{T} \cap \overline{C_N(R_\varepsilon)}$ is a Sylow p-subgroup of $\overline{C_N(R_\varepsilon)}$; but by 5.6.1 we get $N_{T \cdot H}(R_\varepsilon) = T \cdot C_H(R_\varepsilon)$ and therefore $C_{T \cdot H}(R_\varepsilon) = C_T(R) \cdot C_H(R)$ (since $C_T(R_\varepsilon) = C_T(R)$ by Proposition 5.5), whence $\bar{T} \cap \overline{C_N(R_\varepsilon)} = \overline{C_T(R)}$. Hence, setting as in 5.6 (notice that $\mathrm{Ker}(\bar{\eta}) = \overline{C_N(Q_\delta)}$)

$$\tilde{\bar{N}} = \bar{N}/\overline{C_N(R_\varepsilon)} \qquad \text{and} \qquad \tilde{\bar{N}}' = \bar{N}'/\overline{C_{N'}(R'_{\varepsilon'})} \tag{7.7.4}$$

and considering respectively $\bar{N}$ and $\bar{N}'$, endowed with the actions on the sets $\tilde{\bar{N}}$ and $\tilde{\bar{N}}'$ induced by left multiplication, as $\tilde{\bar{N}}$- and $\tilde{\bar{N}}'$-*permutation groups* (cf. 5.6), it follows from Theorem 5.8 and conditions 7.3.1 and 7.7.1 that if $T_\psi \subset P_\gamma$ we have

$$\begin{aligned}\bar{\sigma}(\overline{C_N(R_\varepsilon)}) &= \bar{\sigma}(\overline{C_T(R)}) \cdot \bar{\sigma}(O^p(C_{\bar{N}}(\bar{R}, \tilde{\bar{N}}))) \\ &= \overline{\tau(C_T(R))} \cdot O^p(C_{\bar{N}'}(\bar{R}', \tilde{\bar{N}}')) \subset \overline{C_{N'}(R'_{\varepsilon'})} .\end{aligned}$$

Let $(R_\varepsilon, y, P_\gamma)_{\bar{N}}$ be an $(N, \bar{N})$-exomorphism; now we will prove that the $\bar{N}'$-permutation group homomorphism (cf. 2.5) from $(R', \bar{N}')$ to $(P', \bar{N}')$ mapping $\tau(u)$ upon $\tau(u^y)$ for any $u \in R$ and $\bar{\sigma}(\bar{n})$ upon $\bar{\sigma}(\bar{y}^{-1}\bar{n})$ for any $\bar{n} \in \bar{N}$ determines an element of $E_{N', \bar{N}'}(R'_{\varepsilon'}, P'_{\gamma'})$; that is, we will show that

7.7.5. there is $y' \in N'$ such that $(R'_{\varepsilon'})^{y'} \subset P'_{\gamma'}$, $\bar{\sigma}(\bar{y}) = \bar{y}'$ and $\tau(u^y) = \tau(u)^{y'}$ for any $u \in R$.

We may assume that P_γ contains a defect pointed group of $R \cdot C_N(R_\varepsilon)_\mu$; indeed, anyway we have $n \in N$ such that P_γ contains a defect pointed group of $(R \cdot C_N(R_\varepsilon)_\mu)^n$; if we assume that 7.7.5 holds for any element of $E_{N, \bar{N}}((R_\varepsilon)^n, P_\gamma)$ and denote by $R''_{\varepsilon''}$ the local pointed group on B' such that $Q'_{\delta'} \subset R''_{\varepsilon''} \subset P'_{\gamma'}$ and $R'' = \tau(R^n)$, then there are firstly $n' \in N'$ such that $(R''_{\varepsilon''})^{n'^{-1}} \subset P_{\gamma'}$, $\bar{\sigma}(\bar{n}) = \bar{n}'$ and $\tau(u) = \tau(u^n)^{n'^{-1}}$ for any $u \in R$, whence $R''_{\varepsilon''} = (R'_{\varepsilon'})^{n'}$ (cf. Prop. 5.5), and secondly $y' \in N'$ such that $(R'_{\varepsilon'})^{y'} = (R''_{\varepsilon''})^{n'^{-1}y'} \subset P'_{\gamma'}$, $\bar{\sigma}(\bar{n}^{-1}\bar{y}) = \bar{n}'^{-1}\bar{y}'$ and $\tau(u^y) = \tau(u^n)^{n'^{-1}y'}$ for any $u \in R$.

Consider the N_β-biexomorphism $(R_\varepsilon, 1, y, P_\gamma)$; by hypothesis, there are finite ordered families $\{x_i\}_{i \in I}$ of elements of X and $\{(R_\varepsilon, m_i, P_\gamma \cap (P_\gamma)^{x_i})\}_{i \in I}$ and $\{(P_\gamma, n_i, P_\gamma)\}_{i \in I}$ of N-exomorphisms (cf. 2.8) such that (cf. 2.15.3)

$$(R_\varepsilon, 1, y, P_\gamma) = \sum_{i \in I} (R_\varepsilon, m_i, P_\gamma \cap (P_\gamma)^{x_i})(P_\gamma \cap (P_\gamma)^{x_i}, 1, x_i, P_\gamma)(P_\gamma, n_i, P_\gamma) .$$

Arguing by induction on $|I|$, we may assume that we have

$$(R_\varepsilon, 1, y, P_\gamma) = (R_\varepsilon, 1, s, P_\gamma) + (R_\varepsilon, m, T_\psi)(T_\psi, 1, x, P_\gamma)(P_\gamma, n, P_\gamma)$$

where $x \in X$, $T_\psi = P_\gamma \cap (P_\gamma)^x$, $s = mn$ and $y = mxn$, and that there is $s' \in N'$ such that $(R'_{\varepsilon'})^{s'} \subset P'_{\gamma'}$, $\bar{\sigma}(\bar{s}) = \bar{s}'$ and $\tau(u^s) = \tau(u)^{s'}$ for any $u \in R$ (cf. 7.7.5). Indeed, if i is the last element of I, we certainly may assume that $y = m_i x_i n_i$ and, setting $s = m_i n_i$, it follows from the induction hypothesis that $s = ztv$ where $z \in C_N(R_\varepsilon)$, $v \in P$ and

$(R_\varepsilon, t, P_\gamma)_{\bar N}$ is an $(N, \bar N)$-exomorphism fulfilling 7.7.5 (cf. 2.8); thus, there is $t' \in N'$ such that $(R'_{\varepsilon'})^{t'} \subset P'_{\gamma'}$, $\bar\sigma(\bar t) = \bar t'$ and $\tau(u^t) = \tau(u)^{t'}$ for any $u \in R$; but by 7.7.3 we have $\bar\sigma(\bar z) = \bar z'$ for some $z' \in C_{N'}(R'_{\varepsilon'})$; so, it suffices to set $s' = z't'\tau(v)$.

On the other hand, by condition 7.7.2 there is $x' \in N_{G'}(Q'_{\delta'})$ such that $P'_{\gamma'} \cap (P'_{\gamma'})^{x'} = (P'_{\gamma'} \cap (P'_{\gamma'})^{x'})^{x'}$, $\overline{P' \cap P'^{x'}} = \bar P' \cap \bar P'^{\bar x'}$, $\bar\sigma(\bar x) = \bar x'$ and $\tau(u^x) = \tau(u)^{x'}$ for any $u \in T$; notice that, setting $T'_{\psi'} = P'_{\gamma'} \cap (P'_{\gamma'})^{x'}$, we have

$$\overline{\tau(T)} = \bar\sigma(\overline{P \cap P^x}) = \bar\sigma(\bar P \cap \bar P^{\bar x}) = \bar P' \cap \bar P'^{\bar x'} = \bar T'$$

and therefore $\tau(T) = T'$ (cf. Prop. 5.3). Similarly, there is $n' \in N_{G'}(P'_{\gamma'})$ such that $\bar\sigma(\bar n) = \bar n'$ and $\tau(u^n) = \tau(u)^{n'}$ for any $u \in P$. Set $m' = s'n'^{-1}$ and $y' = m'x'n'$; we claim that $y' \in N'$ fulfils 7.7.5. Clearly $\bar\sigma(\bar m) = \bar m'$ and therefore $\bar\sigma(\bar y) = \bar y'$. Moreover, as $(R'_{\varepsilon'})^{m'} \subset P'_{\gamma'}$ and $T'_{\psi'} \subset P'_{\gamma'}$, by Proposition 5.5 we have $(R'_{\varepsilon'})^{m'} \subset T'_{\psi'}$ (since $R^m \subset T$ implies $\tau(R^m) \subset \tau(T) = T'$, and so $\bar R'^{\bar m'} = \bar\sigma(\overline{R^m}) = \overline{\tau(R^m)} \subset \bar T'$), and therefore we get

$$\tau(u^y) = \tau(u^{mx})^{n'} = \tau(u^m)^{x'n'} = \tau(u)^{y'}\ .$$

Let X' be the set of $x' \in N_{G'}(Q'_{\delta'})$ fulfilling the following conditions:

7.7.6. We have $P'_{\gamma'} \cap (P'_{\gamma'})^{x'} = (P'_{\gamma'} \cap (P'_{\gamma'})^{x'})^{x'}$ *and* $\overline{P' \cap P'^{x'}} = \bar P' \cap \bar P'^{\bar x'}$.

7.7.7. There is $x \in X$ *such that* $P_\gamma \cap (P_\gamma)^x = (P_\gamma \cap (P_\gamma)^x)^x$, $\overline{P \cap P^x} = \bar P \cap \bar P^{\bar x}$, $\bar\sigma(\bar x) = \bar x'$ *and* $\tau(u^x) = \tau(u)^{x'}$ *for any* $u \in \tau^{-1}(P' \cap P'^{x'})$.

Now it suffices to prove that X' contains $N_{G'}(P'_{\gamma'})$ and that the family $\{(P'_{\gamma'} \cap (P'_{\gamma'})^{x'}, 1, x', P'_{\gamma'})\}_{x' \in X'}$ is a generator set of the $N'_{\beta'}$-biexomorphisms; indeed, in that case, we finish the proof invoking symmetry since by 7.7.7 the set X' fulfils condition 7.7.2 with respect to $\bar\sigma^{-1}$ and τ^{-1}. By Theorem 5.8 and conditions 7.3.1 and 7.7.1 we have (cf. 7.7.4)

$$E_N(P_\gamma) = E_{\bar N,\, \tilde{\bar N}}(\bar P) = E_{\bar N',\, \tilde{\bar N}'}(\bar P') = E_{N'}(P'_{\gamma'})$$

and

$$\bar\sigma(\overline{C_N(P_\gamma)}) = \bar\sigma(\overline{Z(P)}) \cdot \bar\sigma(\overline{O^p(C_N(P_\gamma))}) = \overline{\tau(Z(P))} \cdot O^p(C_{\bar N'}(\bar P', \tilde{\bar N}')) = \overline{C_{N'}(P'_{\gamma'})}\ .$$

Hence, for any $n' \in N_{G'}(P'_{\gamma'})$ there is $n \in N_G(P_\gamma)$ such that $\tau(u)^{n'} = \tau(u^n)$ for any $u \in P$, and by 7.7.2 there is $z' \in C_{N'}(P'_{\gamma'})$ such that $\bar\sigma(\bar n) = \overline{z'n'}$; since $\bar z' = \bar\sigma(\bar z)$ for some $z \in C_N(P_\gamma)$, the element $x = z^{-1}n \in X$ fulfils condition 7.7.7 with respect to n' and therefore $n' \in X'$. Moreover, by Corollary 5.11 the family $\{(\bar P \cap \bar P^{\bar x}, 1, \bar x, \bar P)^{\sim}_{\bar N}\}_{x \in X}$ is a generator set of the $(\bar N, \tilde{\bar N})$-biexomorphisms, and therefore the family $\{(\bar P' \cap \bar P'^{\bar\sigma(\bar x)}, 1, \bar\sigma(\bar x), \bar P')^{\sim}_{\bar N'}\}_{x \in X}$ is a generator set of the $(\bar N', \tilde{\bar N}')$-biexomorphisms; but it follows easily from 7.7.2 that $\bar\sigma(\bar X) \subset \bar X'$; consequently, once again by Corollary 5.11, the family $\{(P'_{\gamma'} \cap (P'_{\gamma'})^{x'}, 1, x', P'_{\gamma'})\}_{x' \in X'}$ is a generator set of the $N'_{\beta'}$-biexomorphisms.

Remark 7.8. Actually, since we may assume that for any $x \in X$ the N_β-biexomorphism $(P_\gamma \cap (P_\gamma)^x, 1, x, P_\gamma)$ is irreducible (cf. 2.16.1), a careful reading of our proof shows that in condition 7.7.2 we may replace $\bar\sigma(\bar x) = \bar x'$ by $\bar\sigma(\bar x) \in \bar\sigma(C_N(P_\gamma \cap (P_\gamma)^x))\bar x'$.

8. Source algebra extensions of extensions of nilpotent blocks

8.1. In this section we will prove our main results, namely Theorems 1.8 and 1.12. Actually, in the proof of Theorem 1.12, we will give an explicit description of an $\mathcal{O}$-simple interior P-algebra S such that the P-algebra extension isomorphism 1.12.1 holds. We take once again the notation introduced in 1.6 and 1.7.

8.2. First of all we consider the case when G, H and b fulfil the following additional condition:

8.2.1. There is an indecomposable $\mathcal{O}G$-module M such that $\bar{b}\cdot \mathrm{Res}_H^G(k\otimes_{\mathcal{O}} M)$ is a simple $kH\bar{b}$-module.

Then A maps onto $\mathrm{End}_{\mathcal{O}}(M)$ and, denoting by $\mathfrak{m}$ the kernel of this map, we set $S_\lambda = A_\lambda/f_\lambda^{-1}(\mathfrak{m})$ for any pointed group K_λ on B (cf. 2.11) and denote by $r_\lambda\colon A_\lambda \to S_\lambda$ the canonical map. Clearly S_λ does not depend on the choice of f_λ in $\tilde{f}_\lambda$ and we consider S_λ, endowed with the group homomorphism $K \to S_\lambda^\times$ determined by $r_\lambda\colon A_\lambda^\times \to S_\lambda^\times$ and the structural group homomorphism $K \to A_\lambda^\times$, as an interior K-algebra, whence as a *K-algebra extension* (cf. 2.9).

8.3. Let R_ε be a local pointed group on B such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$. As $r_\beta(B_\beta) = S_\beta$ (cf. 8.2.1), the homomorphism $r_\varepsilon\colon B_\varepsilon \to S_\varepsilon$ is surjective and we claim that it has a section; precisely, considering $\sum_{u\in R} u\cdot B_\varepsilon$ and $S_\varepsilon R$ (cf. 2.9) respectively endowed with B_ε and $S_\varepsilon Q$ as *R-algebra extensions*, we claim that

8.3.1. There is an R-algebra extension isomorphism $q_\varepsilon\colon S_\varepsilon R \cong \sum_{u\in R} u\cdot B_\varepsilon$ such that $r_\varepsilon(q_\varepsilon(s)) = s$ for any $s\in S_\varepsilon$.

Indeed, it is clear that $\sum_{u\in R} u\cdot B_\varepsilon \cong (\mathcal{O}(R\cdot H))_\varepsilon$ as R-algebra extensions, and it follows from Theorem 6.6 applied to $R\cdot H$, H and b that there is an $\mathcal{O}$-simple R-algebra S such that $SR \cong (\mathcal{O}(R\cdot H))_\varepsilon$ (cf. Prop. 5.3 and Remark 5.4); in particular, r_ε induces a surjective R-algebra homomorphism from SQ onto S_ε and therefore an R-algebra isomorphism $S \cong S_\varepsilon$ (cf. [18], 1.8.1). This proves also that:

8.3.2. The p-group R stabilizes an $\mathcal{O}$-basis of S_ε containing the unity as the unique R-fixed element.

8.4. Consider the *R-algebra extension* $S_\varepsilon^0 \otimes_{\mathcal{O}} A_\varepsilon$; by Theorem 5.3 of [18], R has a unique local point $S_\varepsilon^0 \times \varepsilon$ on $S_\varepsilon^0 \otimes_{\mathcal{O}} B_\varepsilon$ and we denote by

$$\tilde{h}_\varepsilon\colon C_\varepsilon \to S_\varepsilon^0 \underset{\mathcal{O}}{\otimes} A_\varepsilon \tag{8.4.1}$$

an embedded algebra extension associated with $S_\varepsilon^0 \times \varepsilon$, and by D_ε the structural subalgebra of C_ε (i.e. $D_\varepsilon = h_\varepsilon^{-1}(S_\varepsilon^0 \otimes_{\mathcal{O}} B_\varepsilon)$); since we have an interior R-algebra embedding $\tilde{d}_\varepsilon\colon \mathcal{O} \to S_\varepsilon^0 \otimes_{\mathcal{O}} S_\varepsilon$ (cf. [18], 5.7) and an R-algebra extension isomorphism $S_\varepsilon R \cong S_\varepsilon \otimes_{\mathcal{O}} \mathcal{O}R$ (cf. [18], 2.7.1), we have an R-algebra extension embedding $\mathcal{O}R \to S_\varepsilon^0 \otimes_{\mathcal{O}} S_\varepsilon R$ and it follows from 8.3.1 and the uniqueness of the local point $S_\varepsilon^0 \times \varepsilon$ that:

8.4.2. There is an R-algebra isomorphism $D_\varepsilon \cong \mathcal{O}Q$.

Another easy consequence of 8.3.1 is that $\tilde{r}_\varepsilon: B_\varepsilon \to S_\varepsilon$ and therefore $\mathrm{id} \otimes \tilde{r}_\varepsilon: S_\varepsilon^0 \otimes_{\mathcal{O}} B_\varepsilon \to S_\varepsilon^0 \otimes_{\mathcal{O}} S_\varepsilon$ are *covering exomorphisms* of R-algebras (cf. [18], 4.14 which can be extended to G-algebras) and, in particular, the unique local point of R on $S_\varepsilon^0 \otimes_{\mathcal{O}} B_\varepsilon$ maps to the unique local point of R on $S_\varepsilon^0 \otimes_{\mathcal{O}} S_\varepsilon$; hence, it follows from 2.11.2 applied to the composition of the R-algebra extension exomorphisms $\tilde{h}_\varepsilon$ and $\mathrm{id} \otimes \tilde{r}_\varepsilon: S_\varepsilon^0 \otimes_{\mathcal{O}} A_\varepsilon \to S_\varepsilon^0 \otimes_{\mathcal{O}} S_\varepsilon$ that:

8.4.3. There is an R-algebra extension homomorphism $t_\varepsilon: C_\varepsilon \to \mathcal{O}$ such that $\tilde{d}_\varepsilon \circ \tilde{t}_\varepsilon = (\mathrm{id} \otimes \tilde{r}_\varepsilon) \circ \tilde{h}_\varepsilon$.

8.5. As $R_\varepsilon \subset P_\gamma$ there is a unique R-algebra extension embedding $\tilde{f}_\varepsilon^\gamma: A_\varepsilon \to \mathrm{Res}_R^P(A_\gamma)$ such that $\mathrm{Res}_R^P(\tilde{f}_\gamma) \circ \tilde{f}_\varepsilon^\gamma = \tilde{f}_\varepsilon$ (cf. 2.11.3); moreover, since

$$f_\varepsilon^{-1}(\mathfrak{m}) = (f_\varepsilon^\gamma)^{-1}(f_\gamma^{-1}(\mathfrak{m}))\,,$$

the embedding $\tilde{f}_\varepsilon^\gamma$ induces an interior R-algebra embedding $\tilde{g}_\varepsilon^\gamma: S_\varepsilon \to \mathrm{Res}_R^P(S_\gamma)$ such that $\tilde{g}_\varepsilon^\gamma \circ \tilde{r}_\varepsilon = \mathrm{Res}_R^P(\tilde{r}_\gamma) \circ \tilde{f}_\varepsilon^\gamma$. Consider the tensor product of these embeddings

$$\tilde{g}_\varepsilon^\gamma \otimes \tilde{f}_\varepsilon^\gamma: S_\varepsilon^0 \underset{\mathcal{O}}{\otimes} A_\varepsilon \to \mathrm{Res}_R^P(S_\gamma^0 \underset{\mathcal{O}}{\otimes} A_\gamma)\,; \tag{8.5.1}$$

it follows from Proposition 5.5 that R has a unique local point on B_γ and therefore on $S_\gamma^0 \otimes_{\mathcal{O}} B_\gamma$ (cf. [18], Th. 5.3); hence, the composition

$$(\tilde{g}_\varepsilon^\gamma \otimes \tilde{f}_\varepsilon^\gamma) \circ \tilde{h}_\varepsilon: C_\varepsilon \to \mathrm{Res}_R^P(S_\gamma^0 \underset{\mathcal{O}}{\otimes} A_\gamma)$$

is an embedded algebra extension associated with the second point; but by 8.4.2 we have $D_\varepsilon \cong \mathrm{Res}_R^P(D_\gamma)$ and, in particular, $\mathrm{Res}_R^P(\tilde{h}_\gamma)$; $\mathrm{Res}_R^P(C_\gamma) \to \mathrm{Res}_R^P(S_\gamma^0 \otimes_{\mathcal{O}} A_\gamma)$ is an embedded algebra extension associated with the same point; therefore, by 2.11.1 we get:

8.5.2. There is a unique R-algebra extension exoisomorphism $\tilde{h}_\varepsilon^\gamma: C_\varepsilon \cong \mathrm{Res}_R^P(C_\gamma)$ such that $(\tilde{g}_\varepsilon^\gamma \otimes \tilde{f}_\varepsilon^\gamma) \circ \tilde{h}_\varepsilon = \mathrm{Res}_R^P(\tilde{h}_\gamma) \circ \tilde{h}_\varepsilon^\gamma$. In particular $t_\gamma \circ h_\varepsilon^\gamma = t_\varepsilon$.

The last equality follows from $\mathrm{Res}_R^P(\tilde{d}_\gamma) = (\tilde{g}_\varepsilon^\gamma \otimes \tilde{f}_\varepsilon^\gamma) \circ \tilde{d}_\varepsilon$ (cf. [18], 5.7); indeed, we have (cf. 8.4.3 and 8.5.2)

$$\begin{aligned}\mathrm{Res}_R^P(\tilde{d}_\gamma \circ \tilde{t}_\gamma) \circ \tilde{h}_\varepsilon^\gamma &= \mathrm{Res}_R^P((\mathrm{id} \otimes \tilde{r}_\gamma) \circ \tilde{h}_\gamma) \circ \tilde{h}_\varepsilon^\gamma = \mathrm{Res}_R^P(\mathrm{id} \otimes \tilde{r}_\gamma) \circ (\tilde{g}_\varepsilon^\gamma \otimes \tilde{f}_\varepsilon^\gamma) \circ \tilde{h}_\varepsilon \\ &= (\tilde{g}_\varepsilon^\gamma \otimes \tilde{g}_\varepsilon^\gamma) \circ (\mathrm{id} \otimes \tilde{r}_\varepsilon) \circ \tilde{h}_\varepsilon = \mathrm{Res}_R^P(\tilde{d}_\gamma) \circ \tilde{t}_\varepsilon\end{aligned}$$

and we apply 1.3 and 1.5 of [17].

8.6. From now on we set $C_\gamma = C$, $D_\gamma = D$, $S_\gamma = S$, $d_\gamma = d$ and $t_\gamma = t$, and choose $C_\varepsilon = \mathrm{Res}_R^P(C)$ endowed with the embedding $(\tilde{h}_\varepsilon^\gamma)^{-1} \circ \tilde{h}_\varepsilon$, denoted again by $\tilde{h}_\varepsilon$ (cf. 2.11 and 8.5.2); moreover, we denote by $\tau: P \to C^\times$ the structural group homomorphism and it follows easily from 8.4 that

$$D = \sum_{v \in Q} \mathcal{O}\tau(v) \qquad \text{and} \qquad \tau(P) \subset N_{C^\times}(\tau(Q)) \cap (1 + \mathrm{Ker}(t))\,. \tag{8.6.1}$$

In particular $\cdot\ N_{C^\times}(\tau(Q))$ normalizes $D^Q = Z(D)$ and we denote by $\rho: N_{C^\times}(\tau(Q)) \to N_{C^\times}(\tau(Q))/(D^Q)^\times$ the canonical map. Let x be an element of

$N_G(R_\varepsilon)$ and denote by φ the automorphism of R induced by x (i.e. $\varphi(u) = u^x$ for any $u \in R$), and by

$$f_x\colon \mathrm{Res}^G_R(A) \cong \mathrm{Res}_\varphi \mathrm{Res}^G_R(A) \tag{8.6.2}$$

the R-algebra extension isomorphism mapping $a_0 \in A$ on a_0^x; as x normalizes R_ε and $\mathfrak{m}$, and R has a unique local point on $S_\varepsilon^0 \otimes_{\mathcal{O}} B_\varepsilon$ (cf. 8.4), it follows from 2.11.1 applied twice that:

8.6.3. There are unique R-algebra extension exoisomorphisms

$\tilde{f}_x^\varepsilon\colon A_\varepsilon \cong \mathrm{Res}_\varphi(A_\varepsilon)$, $\tilde{g}_x^\varepsilon\colon S_\varepsilon \cong \mathrm{Res}_\varphi(S_\varepsilon)$ *and* $\tilde{h}_x^\varepsilon\colon \mathrm{Res}^P_R(C) \cong \mathrm{Res}_\varphi \mathrm{Res}^P_R(C)$ *such that*

$$\mathrm{Res}_\varphi(\tilde{f}_\varepsilon)\circ\tilde{f}_x^\varepsilon = \tilde{f}_x\circ\tilde{f}_\varepsilon,\ \tilde{g}_x^\varepsilon\circ\tilde{r}_\varepsilon = \mathrm{Res}_\varphi(\tilde{r}_\varepsilon)\circ\tilde{f}_x^\varepsilon \text{ and } \mathrm{Res}_\varphi(\tilde{h}_\varepsilon)\circ\tilde{h}_x^\varepsilon = (\tilde{g}_x^\varepsilon \otimes \tilde{f}_x^\varepsilon)\circ\tilde{h}_\varepsilon .$$

But clearly $\mathrm{Res}^R_1(\tilde{f}_x) = \tilde{\mathrm{id}}$ as $\mathcal{O}$-algebra exomorphisms, and therefore we get successively $\mathrm{Res}^R_1(\tilde{f}_x^\varepsilon) = \tilde{\mathrm{id}}$, $\mathrm{Res}^R_1(\tilde{g}_x^\varepsilon) = \tilde{\mathrm{id}}$ (*a priori* true since S_ε is $\mathcal{O}$-simple) and $\mathrm{Res}^R_1(\tilde{h}_x^\varepsilon) = \tilde{\mathrm{id}}$ as $\mathcal{O}$-algebra exomorphisms (cf. [17], 1.5); that is, the maps f_x^ε, g_x^ε and h_x^ε are just conjugation by suitable elements of $A_\varepsilon^\times$, $S_\varepsilon^\times$ and $C^\times$, and the following proposition exhibits a particular choice of them.

Proposition 8.7. *With the notation and hypothesis above*:

8.7.1. There is $a \in A_\varepsilon^\times$ such that $f_\varepsilon(a) \in xB^R$ and $f_x^\varepsilon(a_0) = a_0^a$ for any $a_0 \in A_\varepsilon$.

8.7.2. There is $c \in C^\times$ such that $h_x^\varepsilon(c) \in (r_\varepsilon(a)^{-1} \otimes a)(S_\varepsilon^0 \otimes_{\mathcal{O}} B_\varepsilon)^Q$ and $n_x^\varepsilon(c_0) = c_0^c$ for any $c_0 \in C$.

Remark 8.8. In particular $aB_\varepsilon = f_\varepsilon^{-1}(xB)$ and $cD = h_\varepsilon^{-1}(S_\varepsilon^0 \otimes aB_\varepsilon)$. Moreover, notice that $\tau(u)^c = h_x^\varepsilon(\tau(u)) = \tau(u^x)$ for any $u \in R$.

Proof. Set $f_\varepsilon(1) = i$; since i and i^x belong to ε there is $b' \in (B^R)^\times$ such that $i^{xb'} = i$, and therefore there is a unique $a \in A_\varepsilon^\times$ such that $f_\varepsilon(a) = xb'i$ (since $i(xb')^{-1}xb'i = i$); moreover, denoting by $f_a\colon A_\varepsilon \to A_\varepsilon$ the inner $\mathcal{O}$-algebra automorphism induced by a (i.e. $f_a(a_0) = a_0^a$ for any $a_0 \in A_\varepsilon$) we have

$$f_\varepsilon(f_a(a_0)) = (xb')^{-1} f_\varepsilon(a_0) xb' = f_x(f_\varepsilon(a_0))^{b'}$$

(since $if_\varepsilon(a_0)i = f_\varepsilon(a_0)$), which proves that f_a is an R-algebra extension homomorphism and that $\tilde{f}_a = \tilde{f}_x^\varepsilon$ (cf. 8.6.3).

Similarly, set $h_\varepsilon(1) = j$; since j and $(g_x^\varepsilon \otimes f_x^\varepsilon)(j)$ belong to $S_\varepsilon^0 \times \varepsilon$ (the unique local point of R on $S_\varepsilon^0 \otimes_{\mathcal{O}} B_\varepsilon$) and we may assume that $(g_x^\varepsilon \otimes f_x^\varepsilon)(j) = j^{r_\varepsilon(a)^{-1} \otimes a}$, there is $d' \in ((S_\varepsilon^0 \otimes_{\mathcal{O}} B_\varepsilon)^R)^\times$ such that $j = j^{(r_\varepsilon(a)^{-1} \otimes a)d'}$ and therefore there is a unique $c \in C^\times$ such that $h_\varepsilon(c) = (r_\varepsilon(a)^{-1} \otimes a)d'j$; as above, the inner $\mathcal{O}$-algebra automorphism induced by c is an R-algebra extension homomorphism belonging to $\tilde{h}_x^\varepsilon$.

Corollary 8.9. *With the notation and hypothesis above, assume that $R_\varepsilon = Q_\delta$. There is a unique group homomorphism*

$$\omega\colon N_G(Q_\delta) \to N_{C^\times}(\tau(Q))/(D^Q)^\times \tag{8.9.1}$$

mapping $x \in N_G(Q_\delta)$ upon $\rho(c)$ and we have $\mathrm{Ker}(\omega) = C_H(Q)$.

Proof. By 8.7.2 we have $\tau(v)^c = h_x^\delta(\tau(v)) = \tau(v^x)$ for any $v \in Q$ and therefore $c \in N_{C^\times}(\tau(Q))$. Moreover, if $x' \in N_G(Q_\delta)$, $a' \in A_\delta^\times$ and $c' \in C^\times$ is another system fulfilling the conditions above (i.e. $f_\delta(a') \in x'B^Q$, $f_{x'}^\delta(a_0) = a_0^{a'}$ for any $a_0 \in A_\delta$, $h_\delta(c') \in (r_\delta(a')^{-1} \otimes a')(S_\delta^0 \otimes_{\mathcal{O}} B_\delta)^Q$ and $h_{x'}^\delta(c_0) = c_0^{c'}$ for any $c_0 \in C$), then we have

$$f_\delta(aa') \in xB^Q x'B^Q = xx'B^Q \qquad \text{and} \qquad (f_{x'}^\delta \circ f_x^\delta)(a_0) = a_0^{aa'}$$

for any $a_0 \in A_\delta$, and similarly

$$h_\delta(cc') \in (r_\delta(aa')^{-1} \otimes aa')(S_\delta^0 \underset{\mathcal{O}}{\otimes} B_\delta)^Q \qquad \text{and} \qquad (h_{x'}^\delta \circ h_x^\delta)(c_0) = c_0^{cc'}$$

for any $c_0 \in C$. But it follows easily from 8.6.3 that $\tilde{f}_{x'}^\delta \circ \tilde{f}_x^\delta = \tilde{f}_{xx'}^\delta$ and $\tilde{h}_{x'}^\delta \circ \tilde{h}_x^\delta = \tilde{h}_{xx'}^\delta$, and therefore we may assume that $f_{xx'}^\delta(a_0) = a_0^{aa'}$ and $h_{xx'}^\delta(c_0) = c_0^{cc'}$, which proves that $\rho(c)$ does not depend on the choices of a and c (since $x' = x^{-1}$ forces $cc' \in D^Q$), and that the map ω assigning to x the element $\rho(c) \in N_{C^\times}(\tau(Q))/(D^Q)^\times$ is a group homomorphism.

Moreover, if $x \in C_H(Q)$ we get necessarily $a \in B_\delta^Q$ and $c \in D^Q$ (cf. 8.7.1. and 8.7.2). Conversely, if $x \notin C_H(Q)$ either $x \notin H$ or $x \notin C_G(Q)$; in the first case, we have $B \cap xB = \varnothing$ and we get successively $B_\delta \cap aB_\delta = \varnothing$ and $D \cap cD = \varnothing$ (cf. Remark 8.8); in the second case, since $h_x^\delta(\tau(v)) = \tau(v^x)$ for any $v \in Q$, we have $c \notin C^Q$ (cf. 8.6.1); in both cases $\rho(c) \neq 1$.

8.10. Come back to the general situation above (cf. 8.6). Since $N_G(R_\varepsilon) \subset N_G(Q_\delta)$ (cf. 5.5.1), we may consider (cf. 8.6.3)

$$\tilde{f}_x^\delta \colon A_\delta \cong \mathrm{Res}_\psi(A_\delta), \quad \tilde{g}_x^\delta \colon S_\delta \cong \mathrm{Res}_\psi(S_\delta) \quad \text{and} \quad \tilde{h}_x^\delta \colon \mathrm{Res}_Q^P(C) \cong \mathrm{Res}_\psi \mathrm{Res}_Q^P(C)$$

where ψ denotes the automorphism of Q induced by x, and it follows easily from 2.11.1 and 2.11.3 applied twice that

$$\mathrm{Res}_Q^R(\tilde{f}_x^\varepsilon) \circ \tilde{f}_\delta^\varepsilon = \tilde{f}_\delta^\varepsilon \circ \tilde{f}_x^\delta, \quad \mathrm{Res}_Q^R(\tilde{g}_x^\varepsilon) \circ \tilde{g}_\delta^\varepsilon = \tilde{g}_\delta^\varepsilon \circ \tilde{g}_x^\delta \quad \text{and} \quad \mathrm{Res}_Q^R(\tilde{h}_x^\varepsilon) = \tilde{h}_x^\delta \qquad (8.10.1)$$

where $\tilde{g}_\delta^\varepsilon \colon S_\delta \to \mathrm{Res}_Q^R(S_\varepsilon)$ is the embedding induced by $\tilde{f}_\delta^\varepsilon$ (cf. 8.2). Now we are able to compare $\omega(x)$ with $\rho(c)$ (where a and c have been chosen from R_ε!).

Corollary 8.11. *With the notation and hypothesis above, we have $\omega(x) = \rho(c)$. In particular $\omega(u) = \rho(\tau(u))$ for any $u \in P$.*

Proof. By our choices of C_ε and C_δ (cf. 8.6), and by 2.11.1 we have $\mathrm{Res}_Q^R(\tilde{h}_\varepsilon) = (\tilde{g}_\delta^\varepsilon \otimes \tilde{f}_\delta^\varepsilon) \circ \tilde{h}_\delta$ and therefore it follows from Remark 8.8 that

$$cD = h_\varepsilon^{-1}(S_\varepsilon^0 \otimes f_\varepsilon^{-1}(xB)) = h_\delta^{-1}(S_\delta^0 \otimes f_\delta^{-1}(xB))$$

(notice that we may assume $f_\delta = f_\varepsilon \circ f_\delta^\varepsilon$ and $h_\varepsilon = (g_\delta^\varepsilon \otimes f_\delta^\varepsilon) \circ h_\delta$). On the other hand, by the definition of ω and Proposition 8.7, there is $c' \in C^\times$ such that $\rho(c') = \omega(x)$, $c'D = h_\delta^{-1}(S_\delta^0 \otimes f_\delta^{-1}(xB))$ (cf. Remark 8.8) and $h_x^\delta(c_0) = c_0^{c'}$ for any $c_0 \in C$. Hence $cD = c'D$ and, since $\mathrm{Res}_Q^R(\tilde{h}_x^\varepsilon) = \tilde{h}_x^\delta$ (cf. 8.10.1), there is $d \in (D^Q)^\times$ such that $c'dc^{-1} \in Z(C)$; therefore $c^{-1}c'd$ belongs to $(D^Q)^\times$ and so $\rho(c) = \rho(c') = \omega(x)$. In particular, if $x \in R$ we may choose firstly a equal to the structural image of x in $A_\varepsilon^\times$ and secondly $c = \tau(x)$, which proves the last equality (setting $R_\varepsilon = P_\gamma$).

8.12. Set $F = \mathrm{Im}(\omega)$, $I = \rho^{-1}(F) \cap (1 + \mathrm{Ker}(t))$, $J = I \cap (D^Q)^\times$ and $Z = Z(Q)$. By Corollary 8.9 we have $F \cong N_G(Q_\delta)/C_H(Q)$; therefore, on one hand $\omega(P)$ is a Sylow p-subgroup of F (cf. Prop. 5.3) and, on the other hand, by 1.7.2 we have a group homomorphism

$$\pi\colon I \to \bar{N} \tag{8.12.1}$$

mapping $c \in I$ upon $\bar{x}$ for some $x \in N_G(Q_\delta)$ such that $\rho(c) = \omega(x)$ (i.e. $\overline{\omega^{-1}(\rho(c))} = \{\pi(c)\}$). By 8.6.1 and Corollary 8.11 we have

$$\tau(P) \subset I \qquad \text{and} \qquad J \cap \tau(P) = \tau(Z)\,. \tag{8.12.2}$$

Moreover, notice that J is a p'-divisible abelian group (cf. 8.6.1). The following statement is the key step in our proof of Theorems 1.8 and 1.12.

Theorem 8.13. *With the notation and hypothesis above, there is a finite subgroup L of I such that:*

8.13.1. We have $\tau(P) \subset L$, $I = J \cdot L$ and $\tau(Z) = J \cap L$.

8.13.2. Considering the $\mathcal{O}$-algebra $\mathcal{O}L$ endowed with the group homomorphism $\tau\colon P \to (\mathcal{O}L)^\times$ and the subalgebra $\mathcal{O}(\tau(Q))$ as a P-algebra extension, the inclusion $L \subset C^\times$ induces a P-algebra extension isomorphism $\mathcal{O}L \cong C$.

8.13.3. The group L and the group homomorphisms $\pi\colon L \to \bar{N}$ and $\tau\colon P \to L$ fulfil conditions 1.8.1 and 1.8.2.

Proof. Setting $\bar{I} = I/\tau(Z)$ and $\bar{J} = J/\tau(Z)$, and denoting by $\bar{\rho}\colon \bar{I} \to F$ the group homomorphism determined by ρ, we have the following exact sequence

$$1 \to \bar{J} \to \bar{I} \xrightarrow{\bar{\rho}} F \to 1 \tag{8.13.4}$$

and, since ω induces a group isomorphism $P/Z \cong \omega(P)$, τ induces a group homomorphism $\bar{\tau}\colon \omega(P) \to \bar{I}$ such that $\bar{\rho}(\bar{\tau}(\omega(u))) = \omega(u)$ for any $u \in P$ (cf. Cor. 8.11); that is, $\bar{\tau}$ fulfils condition 4.7.1 and we will prove that $\bar{\tau}$ fulfils also condition 4.7.2 with respect to the elements $f \in F$ such that $\omega(P) \cap \omega(P)^f = (\omega(P) \cap \omega(P)^f)^f$. Let f be such an element and R the subgroup of P containing Q such that $\omega(R) = \omega(P) \cap \omega(P)^f$ (notice that $\omega(Q)$ is normal in F); by Proposition 5.5, there is a unique $\varepsilon \in \mathcal{L}\mathcal{P}_B(R)$ such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$ and, if λ is the unique point of $R \cdot C_H(Q)$ on B such that $Q_\delta \subset R \cdot C_H(Q)_\lambda$ (cf. 2.16.2), R_ε is a defect pointed group of $R \cdot C_H(Q)_\lambda$ (cf. 5.5.2). Hence, by the Frattini argument, there is $x \in N_G(R_\varepsilon)$ such that $\omega(x) = f$, and by Corollary 8.11 there is $c \in N_{C^\times}(\tau(Q))$ such that $\rho(c) = \omega(x) = f$ and $\tau(u^x) = \tau(u)^c$ for any $u \in R$ (cf. Remark 8.8); as we may assume that $c \in I$ (up to replacing c by $t(c)^{-1}c$), we get $\bar{\rho}(\bar{c}) = f$ and $\bar{\tau}(\omega(u))^{\bar{c}} = \bar{\tau}(\omega(u)^f)$ for any $u \in R$, where $\bar{c}$ is the image of c in $\bar{I}$.

It follows now from 2.17.1 and Corollary 4.7 that there is a group homomorphism $\bar{\sigma}\colon F \to \bar{I}$ such that $\bar{\rho} \circ \bar{\sigma} = \mathrm{id}_F$ and $\bar{\sigma}(\omega(u)) = \bar{\tau}(\omega(u))$ for any $u \in P$. We claim that the inverse image L of $\mathrm{Im}(\bar{\sigma})$ in I fulfils conditions 8.13.1 to 8.13.3 above. Condition 8.13.1 follows easily from the properties of $\bar{\sigma}$. Moreover, we have $\pi(L) = \bar{N} \cong N_G(Q_\delta)/Q \cdot C_H(Q)$ (cf. 8.12.1, 8.13.1 and 1.7.2) and, if $\{x_{\bar{n}}\}_{\bar{n} \in \bar{N}}$ and

$\{y_{\bar{n}}\}_{\bar{n}\in\bar{N}}$ are respectively sets of representatives for $\bar{N}$ in $N_G(Q_\delta)$ and in L, it is clear that $\omega(x_{\bar{n}}) = \rho(y_{\bar{n}})$ for any $\bar{n}\in\bar{N}$ (cf. 8.12.1), and that $A_\beta = \oplus_{\bar{n}\in\bar{N}} x_{\bar{n}}B_\beta$ (cf. 1.7); then it follows from Remark 8.8 that

$$y_{\bar{n}}D = h_\delta^{-1}(S_\delta \otimes f_\delta^{-1}(x_{\bar{n}}B_\beta))$$

for any $\bar{n}\in\bar{N}$ and therefore we get $C = \oplus_{\bar{n}\in\bar{N}} y_{\bar{n}}D$, which proves condition 8.13.2. On the other hand $L\cap \mathrm{Ker}(\pi) = \tau(Q)$ (since $L\cap J = \tau(Z)$), $\mathrm{Ker}(\tau) = \{1\}$ (cf. 8.6.1 and Cor. 8.11) and $\tau(P)$ is a Sylow p-subgroup of L; in particular, in 7.2 we may assume that $G' = L$, $H' = \tau(Q)$ and $b' = 1$, and in 7.3 we may consider the group isomorphisms $\bar{N}\cong L/\tau(Q)$ and $P\cong\tau(P)$ induced respectively by π and τ; then it is quite clear that conditions 7.3.1 and 7.3.2 imply conditions 1.8.1 and 1.8.2. So, it suffices to prove that A_α and $\mathcal{O}L$ are *projectively equivalent* through the group isomorphisms $\bar{N}\cong L/\tau(Q)$ and $P\cong\tau(P)$ above.

Since $\pi(\tau(u)) = \bar{u}$ for any $u\in P$ (cf. 8.12.1 and Cor. 8.11) and $\pi(C_L(\tau(Q))) = \overline{C_N(Q_\delta)}$ (cf. 8.12.1 and Remark 8.8), conditions 7.3.1 and 7.7.1 hold and, by 7.6 and Theorem 7.7, it suffices to prove that condition 7.7.2 holds with respect to the elements $x\in N_G(Q_\delta)$ such that $\overline{P\cap P^x} = \bar{P}\cap\bar{P}^{\bar{x}}$ and $P_\gamma\cap(P_\gamma)^x = (P_\gamma\cap(P_\gamma)^x)^x$. Let x be such an element and set $R_\varepsilon = P_\gamma\cap(P_\gamma)^x$; by Corollary 8.11 and Remark 8.8 there is $c\in N_{C^x}(\tau(Q))$ such that $\omega(x) = \rho(c)$ and $\tau(u^x) = \tau(u)^c$ for any $u\in R$. Hence, since we may assume that $c\in I$ (up to replacing c by $t(c)^{-1}c$) and we have $I = J\cdot L$ (cf. 8.13.1), we get $c = zy$ for some $y\in L$ and some $z\in J$; moreover, as c normalizes $\tau(R)$ and we have $J\cap L = \tau(Z)\subset\tau(R)$, y and therefore z still normalize $\tau(R)$ and, since $N_J(\tau(R)) = C_J(\tau(R))\cdot\tau(N_Z(R))$ by Lemma 8.15 below, we may assume that z centralizes $\tau(R)$, whence $\rho(z) = 1$. In that case, we get $\pi(y) = \bar{x}$ (since $\rho(y) = \rho(c) = \omega(x)$) and $\tau(u^x) = \tau(u)^y$ for any $u\in R$, and then it is quite clear that $\bar{R} = \bar{P}\cap\bar{P}^{\bar{x}}$ implies both $\tau(R) = \tau(P)\cap\tau(P)^y$ and that the image of $\tau(R)$ in $L/\pi(Q)$ is the intersection of the images of $\tau(P)$ and $\tau(P)^y$.

Remark 8.14. The existence of $\bar{\sigma}\colon F\to\bar{I}$ above follows also from a result of Oliver [11]. Indeed, we may assume that $\mathcal{k}$ is an algebraic closure of the prime field and that $\mathcal{O}$ is totally unramified. In that case Oliver's result implies that $\mathbb{H}^1(\omega(P),\bar{J}) = \{0\}$. On the other hand, the existence of $\bar{\tau}$ shows already that the exact sequence 8.13.4 splits, and the existence of $\bar{\sigma}$ is equivalent to the existence of a complement $\bar{L}$ of $\bar{J}$ in $\bar{I}$ containing $\bar{\tau}(\omega(P))$. But a complement of $\bar{J}$ in $\bar{I}$ contains certainly a complement of $\bar{J}$ in $\bar{J}\cdot\bar{\tau}(\omega(P))$ and Oliver's result forces the last one to be $\bar{\tau}(\omega(P))^{\bar{z}}$ for some $\bar{z}\in\bar{J}$.

The following lemma, quoted above, is a slight generalization of a result by Coleman [5] and our proof employs his argument (see [17] for a different generalization).

Lemma 8.15. *Let K be a group, Q a subgroup of K and R a finite p-subgroup of $N_K(Q)$. We have $N_{(\mathcal{O}Q)^\times}(R) = C_{(\mathcal{O}Q)^\times}(R)\cdot N_Q(R)$. In particular, if $Q\subset R$ then $N_{Z(\mathcal{O}Q)^\times}(R) = C_{Z(\mathcal{O}Q)^\times}(R)\cdot N_{Z(Q)}(R)$.*

Proof. For any $a\in N_{(\mathcal{O}Q)^\times}(R)$ we consider the action of R on $\mathcal{O}Q$ mapping $c\in\mathcal{O}Q$ upon $u^{-1}ca^{-1}ua$ for any $u\in R$; clearly R stabilizes Q (since $Q = K\cap\mathcal{O}Q$ in $\mathcal{O}K$) and

fixes a through this action; so if $a = \sum_{x \in Q} \lambda_x x$ where $\lambda_x \in \mathcal{O}$ for any $x \in Q$, there is $x \in Q$ fixed by R such that $\lambda_x \in \mathcal{O}^\times$ since the augmentation map $\mathcal{O}Q \to \mathcal{O}$ assigns to a an invertible element; that is, we have $a^{-1}ua = x^{-1}ux$ for any $u \in R$ and therefore $a = zx$ for some $z \in C_{(\mathcal{O}Q)^\times}(R)$.

8.16. Henceforth we prove Theorems 1.8 and 1.12. In order to prove the first one and the existence of S in the second, we may assume that $\mathcal{O}$ is totally unramified (cf. [18], 2.13.1) and, in particular, that $\mathcal{O}$ contains a non-trivial root of unity ξ of p-power order only if $p = 2$ and $\xi = -1$. Let $\mathfrak{n}_\delta$ be the unique ideal of B_δ such that $B_\delta/\mathfrak{n}_\delta$ is an $\mathcal{O}$-simple algebra and the determinant of the structural image of any $v \in Q$ is one (cf. [18], Th. 1.6 and 1.8.1), and $\mathfrak{n}_\beta$ the unique ideal of B_β such that $(f_\delta^\beta)^{-1}(\mathfrak{n}_\beta) = \mathfrak{n}_\delta$ (cf. 2.14.1). We claim that:

8.16.1. For any $x \in N$, we have $(\mathfrak{n}_\beta)^x = \mathfrak{n}_\beta$.

Indeed, by 1.7.2 we may assume that $x \in N_G(Q_\delta)$ and, as in 8.6, we denote by f_x^β: $\mathrm{Res}_Q^N(A_\beta) \cong \mathrm{Res}_\varphi \mathrm{Res}_Q^N(A_\beta)$ the $\mathcal{O}$-algebra extension isomorphism mapping $a \in A_\beta$ upon a^x, where φ is the automorphism of Q induced by x; then, by 2.11.1, there is a Q-algebra extension exoisomorphism $\tilde{f}_x^\delta$: $A_\delta \cong \mathrm{Res}_\varphi(A_\delta)$ such that $\mathrm{Res}_\varphi(\tilde{f}_\delta^\beta) \circ \tilde{f}_x^\delta = \tilde{f}_x^\beta \circ \tilde{f}_\delta^\beta$; now the uniqueness of $\mathfrak{n}_\delta$ forces $f_x^\delta(\mathfrak{n}_\delta) = \mathfrak{n}_\delta$, whence we get $f_x^\beta(\mathfrak{n}_\beta) = \mathfrak{n}_\beta$. In particular, it follows easily from 8.16.1 that G stabilizes the isomorphism class of an $\mathcal{O}H\mathrm{Tr}_N^G(b)$-module M such that $k \otimes_{\mathcal{O}} b \cdot M$ is a simple $kH\bar{b}$-module.

8.17. Thus there is always an extension G' of G by an abelian finite group U' such that H lifts to a normal subgroup of G' – still denoted by H – and such that the triple G', H and b fulfils condition 8.2.1. Moreover, denoting respectively by N' and V' the stabilizers of b in G' and in U', we may assume that

$$V' \subset D(N') \cap Z(N') \tag{8.17.1}$$

and that V' is isomorphic to a subgroup of $\mathcal{O}^\times$. As in 1.6, set $A' = \mathcal{O}G'$, consider A' endowed with B as a *G'-algebra extension* and denote by $\bar{N}'$ the quotient N'/H. Clearly α is still the unique point of G' on B such that $H_\beta \subset G'_\alpha$, β is still a point of N' on B and, denoting by P' the Sylow p-subgroup of the inverse image of P in G', γ is still a local point of P' on B such that $Q_\delta \subset P'_\gamma \subset N'_\beta \subset G'_\alpha$ and P'_γ is also a defect pointed group of G'_α. As in 1.7, denote respectively by

$$\tilde{f}_\alpha'': A'_\alpha \to A, \tilde{f}_\beta'': A'_\beta \to \mathrm{Res}_{N'}^{G'}(A) \qquad \text{and} \qquad \tilde{f}_\varepsilon'': A'_\varepsilon \to \mathrm{Res}_{R'}^{G'}(A)$$

embedded algebra extensions associated with G'_α, N'_β and any local pointed group R'_ε on B such that $Q_\delta \subset R'_\varepsilon \subset P'_\gamma$. Notice that, since $A'_\beta \cong \mathcal{O}N'b$ (cf. 1.7), A'_β has an evident $\mathcal{O}V'$-algebra structure and then A'_ε has a unique $\mathcal{O}V'$-algebra structure such that the canonical embedding $\tilde{f}_\varepsilon''^\beta$: $A'_\varepsilon \to \mathrm{Res}_{R'}^{N'}(A'_\beta)$ (cf. 2.11.3) becomes an embedding of *R'-algebra extensions* over $\mathcal{O}V'$; thus, since $A_\beta \cong \mathcal{O}Nb$ (cf. 1.7), we have N- and therefore R-algebra extension isomorphisms, R being the image of R' in P and $\mathcal{O}$ the trivial $\mathcal{O}V'$-algebra,

$$A_\beta = \mathcal{O} \underset{\mathcal{O}V'}{\otimes} A'_\beta \qquad \text{and} \qquad A_\varepsilon = \mathcal{O} \underset{\mathcal{O}V'}{\otimes} A'_\varepsilon \tag{8.17.2}$$

where $\mathcal{O} \otimes_{\mathcal{O}V'} A'_\beta$ and $\mathcal{O} \otimes_{\mathcal{O}V'} A'_\varepsilon$ are respectively endowed with the subalgebras $1 \otimes B_\beta$ and $1 \otimes B_\varepsilon$, and with the evident group homomorphisms.

8.18. Since G', H and b fulfil condition 8.2.1, there is an ideal $\mathfrak{m}'$ of A' such that $A'/\mathfrak{m}'$ is an $\mathcal{O}$-simple algebra and we have

$$bA'b = Bb + b\mathfrak{m}'b\ ,$$

and we may assume that the restriction $U' \to (A'/\mathfrak{m}')^\times$ of the structural homomorphism is injective. As in 8.2, we set $S'_\varepsilon = A'_\varepsilon/f'^{-1}_\varepsilon(\mathfrak{m}')$ considered as an interior R'-algebra, whence as an *R'-algebra extension* (cf. 2.9), and by 8.3.2 R' stabilizes an $\mathcal{O}$-basis of S'_ε containing the unity as the unique R'-fixed element. Notice that the $\mathcal{O}V'$-algebra structure of A'_ε induces an $\mathcal{O}V'$-algebra structure on S'_ε, namely an $\mathcal{O}$-algebra homomorphism

$$r' : \mathcal{O}V' \to \mathcal{O} \tag{8.18.1}$$

which does not depend on the choice of R_ε.

8.19. As in 8.4, we denote by $\tilde{h}'_\varepsilon : C'_\varepsilon \to S'^0_\varepsilon \otimes_{\mathcal{O}} A'_\varepsilon$ an embedded algebra extension associated with the unique local point of R' on $S'^0_\varepsilon \otimes_{\mathcal{O}} B_\varepsilon$, by D'_ε the structural subalgebra of C'_ε and by $t'_\varepsilon : C'_\varepsilon \to \mathcal{O}$ the R'-algebra extension homomorphism 8.4.3. The $\mathcal{O}V'$-algebra structures of A'_ε and S'_ε induce an $\mathcal{O}V'$-algebra structure on $S'^0_\varepsilon \otimes_{\mathcal{O}} A'_\varepsilon$ (through the diagonal map $V' \to V' \times V'$); clearly there is a unique $\mathcal{O}V'$-algebra structure on C'_ε such that $\tilde{h}'_\varepsilon$ becomes an embedding of *R'-algebra extensions* over $\mathcal{O}V'$ and it is easily checked that t'_ε is an $\mathcal{O}V'$-algebra homomorphism. As in 8.6, we set $C'_\gamma = C'$, $D'_\gamma = D'$, $S'_\gamma = S'$, $d'_\gamma = d'$ and $t'_\gamma = t'$, and we denote by $\tau' : V' \cdot P' \to C'^\times$ the structural group homomorphism (both as *P'-algebra extension* and $\mathcal{O}V'$-algebra) and by $\rho' : N_{C'^\times}(\tau'(Q)) \to N_{C'^\times}(\tau'(Q))/(D'^Q)^\times$ the canonical map.

8.20. Now Proposition 8.7 and Corollaries 8.9 and 8.11 apply to our situation, and we denote by

$$\omega' : N_{G'}(Q_\delta) \to N_{C'^\times}(\tau'(Q))/(D'^Q)^\times$$

the group homomorphism 8.9.1; notice that:

8.20.1. For any $v' \in V'$ we have $\omega(v') = \rho'(\tau'(v'))$.

Moreover, as in 8.12 we set $F' = \mathrm{Im}(\omega')$, $I' = \rho'^{-1}(F') \cap (1 + \mathrm{Ker}(t'))$ and $J' = I' \cap \mathrm{Ker}(\rho')$, and denote by $\pi' : I' \to \bar{N}'$ the group homomorphism 8.12.1; notice that (cf. 8.20.1 and Cor. 8.11):

8.20.2. We have $\tau'(V') \subset I'$, $J' \cap \tau'(V') = \{1\}$ and $\pi'(\tau'(v')) = \bar{v}'$ for any $v' \in V'$. Moreover, the complement of $\tau'(V' \cap P')$ in $\tau'(V')$ is the unique complement of $J' \cdot \tau'(V' \cap P')$ in $J' \cdot \tau'(V')$.

Hence, by Theorem 8.13, there is a finite subgroup L' of I' such that:

8.20.3. We have $\tau'(V' \cdot P') \subset L'$, $I' = J' \cdot L'$ and $\tau'(Z) = J' \cap L'$.

8.20.4. Considering the $\mathcal{O}$-algebra $\mathcal{O}L'$ endowed with the group homomorphism $\tau' : V' \cdot P' \to (\mathcal{O}L')^\times$ and the subalgebra $\mathcal{O}(\tau'(Q))$, the inclusion $L' \subset C'^\times$ induces an isomorphism $\mathcal{O}L' \cong C'$ both as P'-algebra extensions and as $\mathcal{O}V'$-algebras.

8.20.5. The group L' and the group homomorphisms $\pi': L' \to \bar{N}'$ and $\tau': P' \to L'$ fulfil conditions 1.8.1 and 1.8.2 with respect to G', H and b.

8.21. We claim that the group $L = L'/\tau'(V')$ and the group homomorphisms $\pi: L \to \bar{N}$ and $\tau: P \to L$ induced respectively by π' and τ' fulfil conditions 1.8.1 and 1.8.2 with respect to G, H and b. Indeed, it is quite clear that condition 1.8.1 with respect to G', H and b implies condition 1.8.1 with respect to G, H and b. Moreover, let R_ε be a local pointed group on B such that $Q_\delta \subset R_\varepsilon \subset P_\gamma$ and denote by R' the inverse image of R in P'. Recall that we consider respectively N and N' as $\bar{N}$- and $\bar{N}'$-*permutation groups* (cf. 1.6), and let us denote by $\tilde{\mathrm{Hom}}_{\bar{N}'}(R', P')_{V'}$ the set of $\bar{N}'$-permutation group exomorphisms $\widehat{(\varphi', f')}$ from R' to P' (cf. 2.5) such that $\varphi'(V' \cap R') \subset V' \cap P'$ and $f'(\bar{n}'\bar{V}') = f'(\bar{n}')\bar{V}'$ for any $\bar{n}' \in \bar{N}'$, which clearly contains $E_{N',\bar{N}'}(\tau'(R'_\varepsilon, P'_\gamma))$ (cf. 2.8); then, we have an evident map

$$\tilde{\mathrm{Hom}}_{\bar{N}'}(R', P')_{V'} \to \tilde{\mathrm{Hom}}_{\bar{N}}(R, P) \tag{8.21.1}$$

and it is clear that $E_{N',\bar{N}'}(R'_\varepsilon, P'_\gamma)$ maps onto $E_{N,\bar{N}}(R_\varepsilon, P_\gamma)$. Similarly, we consider respectively L and L' as $\bar{N}$- and $\bar{N}'$-permutation groups (cf. 1.8.1), we introduce the set of $\bar{N}'$-permutation group exomorphisms $\tilde{\mathrm{Hom}}_{\bar{N}'}(\tau'(R'), \tau'(P'))_{V'}$ containing $E_{L',\bar{N}'}(\tau'(R'), \tau'(P'))$ and we have a map

$$\tilde{\mathrm{Hom}}_{\bar{N}'}(\tau'(R'), \tau'(P'))_{V'} \to \tilde{\mathrm{Hom}}_{\bar{N}'}(\tau(R), \tau(P)) \tag{8.21.2}$$

sending $E_{L',\bar{N}'}(\tau'(R'), \tau'(P'))$ onto $E_{L,\bar{N}}(\tau(R), \tau(P))$. Since the bijections τ_R: $\tilde{\mathrm{Hom}}_{\bar{N}}(R, P) \cong \tilde{\mathrm{Hom}}_{\bar{N}}(\tau(R), \tau(P))$ and $\tau'_{R'}: \tilde{\mathrm{Hom}}_{\bar{N}'}(R', P')_{V'} \cong \tilde{\mathrm{Hom}}_{\bar{N}'}(\tau'(R'), \tau'(P'))_{V'}$ induced respectively by τ and τ' are compatible with the maps 8.21.1 and 8.21.2, condition 1.8.2 with respect to G, H and b follows from condition 1.8.2 with respect to G', H and b. To finish the proof of Theorem 1.8 it suffices to discuss the uniqueness of L, π and τ; this uniqueness follows easily from Proposition 4.9 since conditions 1.8.1 and 1.8.2 imply respectively conditions 4.9.1 and 4.9.2 with respect to any subset of L.

8.22. We will prove now the existence part of Theorem 1.12. Since r' maps injectively V' into $\mathcal{O}^\times$ and p divides neither $\mathrm{rank}_{\mathcal{O}}(S')$ (cf. 8.18) nor $|L': \tau'(P')|$ (cf. 8.20.5 and Remark 1.9), the composition θ' of the structural homomorphism $L' \to \mathrm{Ind}_{\tau'(P')}^{L'}(S'^{\tau'})^\times$, where $S'^{\tau'}$ is the restriction of S' to $\tau'(P')$, with the determinant map maps injectively $\tau'(V' \cap P')$ into $\mathcal{O}^\times$; but, as we assume that $\mathcal{O}$ is totally unramified, $\theta'(\tau'(V' \cap P'))$ has always a complement $\mathcal{U}$ in the group of the roots of unity of $\mathcal{O}$; consequently, $L'' = \theta'^{-1}(\mathcal{U})$ is a complement of $\tau'(V' \cap P')$ in L' and in particular, setting $\tau'(P'') = \tau'(P') \cap L''$, P'' is a complement of $V' \cap P'$ in P', so that $P'' \cong P$. We denote by $\tau'': P \to L''$ the group homomorphism induced by τ' and this isomorphism, and by S'' the restriction of S' to P. Then, by Remark 1.13, it suffices to prove that we have

$$A_\gamma \cong S'' \otimes_{\mathcal{O}} \mathcal{O}_* \hat{L}^0 \tag{8.22.1}$$

as P-algebra extensions.

8.23. First of all, notice that we have the following P-algebra extension embeddings (cf. 8.4 and 8.19)

$$\mathrm{i}\tilde{\mathrm{d}} \otimes \tilde{h}'_\gamma : S' \underset{\mathcal{O}}{\otimes} C' \to S' \underset{\mathcal{O}}{\otimes} S'^0 \underset{\mathcal{O}}{\otimes} A'_\gamma \quad \text{and} \quad \tilde{d}' \otimes \mathrm{i}\tilde{\mathrm{d}} : A'_\gamma \to S' \underset{\mathcal{O}}{\otimes} S'^0 \underset{\mathcal{O}}{\otimes} A'_\gamma \qquad (8.23.1)$$

which are embedded algebra extensions associated with the unique local point of P' on $S' \otimes_{\mathcal{O}} S'^0 \otimes_{\mathcal{O}} B_\gamma$ (cf. 8.4.2 and [18], 5.3.1 extended to P-algebras). Moreover, the $\mathcal{O}V'$-algebra structures of A'_γ, S' and C' induce successively unique $\mathcal{O}V'$-algebra structures on $S' \otimes_{\mathcal{O}} S'^0 \otimes_{\mathcal{O}} A'_\gamma$ and $S' \otimes_{\mathcal{O}} C'$ in such a way that $\tilde{d}' \otimes \mathrm{i}\tilde{\mathrm{d}}$ and $\mathrm{i}\tilde{\mathrm{d}} \otimes \tilde{h}'_\gamma$ become $\mathcal{O}V'$-algebra embeddings. Consequently, by 2.11.1 and 8.20.4 we have an isomorphism $A'_\gamma \cong S' \otimes_{\mathcal{O}} \mathcal{O}L'$ both as *P'-algebra extensions* and as $\mathcal{O}V'$-algebras, and by 8.17.2 we get a P-algebra extension isomorphism, $\mathcal{O}$ being the trivial $\mathcal{O}V'$-algebra,

$$A_\gamma \cong \mathcal{O} \underset{\mathcal{O}V'}{\otimes} \left(S' \underset{\mathcal{O}}{\otimes} \mathcal{O}L' \right). \qquad (8.23.2)$$

8.24. But denoting by V'' the complement of $V' \cap P'$ in V' and considering $\mathcal{O}L''$, endowed with the group homomorphism $P \to (\mathcal{O}L'')^\times$ induced by τ'' and with the image of $\mathcal{O}Q$, as a *P-algebra extension*, it is not difficult to check that we have P-algebra extension isomorphisms

$$S'' \underset{\mathcal{O}}{\otimes} \left(\mathcal{O}'' \underset{\mathcal{O}V''}{\otimes} \mathcal{O}L'' \right) \cong \mathcal{O} \underset{\mathcal{O}V''}{\otimes} \left(S'' \underset{\mathcal{O}}{\otimes} \mathcal{O}L'' \right) \cong \mathcal{O} \underset{\mathcal{O}V'}{\otimes} \left(S' \underset{\mathcal{O}}{\otimes} \mathcal{O}L' \right) \qquad (8.24.1)$$

mapping $s'' \otimes (1 \otimes y'')$ upon $1 \otimes (s'' \otimes y'')$ where $s'' \in S''$, $y'' \in L''$ and $\mathcal{O}''$ denotes the ring $\mathcal{O}$ endowed with the $\mathcal{O}V''$-algebra structure mapping any $v'' \in V''$ upon $r'(v'')^{-1}$ (cf. 8.18.1). Moreover, as

$$bA'b/b\mathfrak{m}'b \cong Bb/(B \cap \mathfrak{m}')b \quad \text{and} \quad \kappa \underset{\mathcal{O}}{\otimes} (Bb/(B \cap \mathfrak{m}')b) \cong B_\beta / J(B_\beta) \qquad (8.24.2)$$

(cf. 8.18), the structural homomorphism $N' \to (bA'b/b\mathfrak{m}'b)^\times$ induces a group homomorphism $\bar{\sigma}' : \bar{N}' \to \hat{\bar{N}}$ (cf. 1.11) mapping any $\bar{v}'' \in \bar{V}''$ upon $\overline{r'(v'')} \in \kappa^\times$ (cf. 8.18.1), and therefore a group homomorphism $\sigma' : L' \to \hat{L}$ such that $\hat{\pi} \circ \sigma' = \bar{\sigma}' \circ \pi'$ where $\hat{L} = \mathrm{Res}_\pi(\hat{\bar{N}})$ (cf. 2.4) and $\hat{\pi} : \hat{L} \to \hat{\bar{N}}$ is the canonical map; now it can be checked that we have a P-algebra extension isomorphism

$$\mathcal{O}'' \underset{\mathcal{O}V''}{\otimes} \mathcal{O}L'' \cong \mathcal{O}_* \hat{L}^0 \qquad (8.24.3)$$

mapping $1 \otimes y''$ upon $\sigma'(y'')^{-1}$ for any $y'' \in L''$. The isomorphism 8.22.1 follows now from the isomorphisms 8.23.2, 8.24.1 and 8.24.3.

8.25. It remains to prove the uniqueness part of Theorem 1.12, without any hypothesis on the ramification of $\mathcal{O}$. First of all, we claim that

8.25.1. if S is an $\mathcal{O}$-simple interior P-algebra such that $A_\gamma \cong S \otimes_{\mathcal{O}} \mathcal{O}_* \hat{L}^0$ then P stabilizes an $\mathcal{O}$-basis of $\mathrm{Ind}_{\tau(P)}^{L}(S^\tau)$ *where S^τ is the restriction of S to $\tau(P)$.*

Indeed, for any $x, y \in L$, it suffices to prove that $\tau(R) = \tau(P) \cap \tau(P)^x \cap \tau(P)^y$ stabilizes an $\mathcal{O}$-basis of $x^{-1} \otimes S^\tau \otimes y$, or equivalently, that R stabilizes an $\mathcal{O}$-basis of

$\mathrm{Res}^{P\times P}_{\delta_{x,y}(R)}(S)$ where $\delta_{x,y}: R \to P \times P$ maps $u \in R$ upon $(u^{x^{-1}}, (u^{y^{-1}})^{-1})$ (we identify R with $\delta_{x,y}(R)$); but, if $\hat{x}$ and $\hat{y}$ lift respectively x and y to $\hat{L}$, it is clear that $\sum_{u\in R} \mathcal{O}\hat{y}\tau(u)\hat{x}^{-1}$ is a direct summand of ${}^{\mathcal{O}}_{*}\hat{L}^0$ isomorphic to $\mathrm{Ind}^{R^x\times R^y}_{\delta_{x,y}(R)}(\mathcal{O})$ as $\mathcal{O}(R^x \times R^y)$-modules and therefore $\mathrm{Ind}^{R^x\times R^y}_{\delta_{x,y}(R)}(\mathrm{Res}^{P\times P}_{\delta_{x,y}(R)}(S))$ is a direct summand of A_y as $\mathcal{O}(R^x \times R^y)$-module; in particular, $\mathrm{Res}^{P\times P}_{\delta_{x,y}(R)}(S)$ is a direct summand of $\mathrm{Res}^{P\times P}_{\delta_{x,y}(R)}(A_y)$, so it is a permutation $\mathcal{O}R$-module.

8.26. Let S and T be $\mathcal{O}$-simple interior P-algebras fulfilling the conditions of Theorem 1.12. By Remark 1.14, we know already that S and T are isomorphic as P-algebras; hence, we have $k \otimes_{\mathcal{O}} S \cong k \otimes_{\mathcal{O}} T$ as *interior* P-algebras (since the lifting of the action of P is unique) and, in particular,

$$k \underset{\mathcal{O}}{\otimes} \mathrm{Ind}^{L}_{\tau(P)}(S^\tau) \cong k \underset{\mathcal{O}}{\otimes} \mathrm{Ind}^{L}_{\tau(P)}(T^\tau) \tag{8.26.1}$$

where S^τ and T^τ are respectively the restrictions of S and T to $\tau(P)$. But it follows from 8.25.1 and from the fact that p does not divide the dimension of both terms in 8.26.1, that the P-algebras $\mathrm{Res}_\tau(\mathrm{Ind}^{L}_{\tau(P)}(S^\tau))$ and $\mathrm{Res}_\tau(\mathrm{Ind}^{L}_{\tau(P)}(T^\tau))$ fulfil conditions 1.3.2 and 1.3.3 in [20] and therefore they are isomorphic (cf. [20], Cor. 3.7). Moreover, since the determinant maps the images of P in $\mathrm{Ind}^{L}_{\tau(P)}(S^\tau)^\times$ and $\mathrm{Ind}^{L}_{\tau(P)}(T^\tau)^\times$ onto $\{1\}$, we get

$$\mathrm{Res}_\tau(\mathrm{Ind}^{L}_{\tau(P)}(S^\tau)) \cong \mathrm{Res}_\tau(\mathrm{Ind}^{L}_{\tau(P)}(T^\tau)) \tag{8.26.2}$$

as interior P-algebras. As the canonical embeddings $S \to \mathrm{Res}_\tau(\mathrm{Ind}^{L}_{\tau(P)}(S^\tau))$ and $T \to \mathrm{Res}_\tau(\mathrm{Ind}^{L}_{\tau(P)}(T^\tau))$ (cf. 2.13.1) are embedded interior P-algebras associated with the unique local point of P on both terms of 8.26.2 (cf. [18], Cor. 5.8), we get finally $S \cong T$ as interior P-algebras.

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On the vanishing of local cohomology modules

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1. Introduction

Let A be a commutative ring, and let I be an ideal of A. The *cohomological dimension of I in A*, denoted by $cd(A, I)$, is the smallest integer n such that the local cohomology modules $H^q_I(M) = 0$ for all A-modules M, and for all $q > n$. (Actually if $H^q_I(A) = 0$ for all $q > n$, then $cd(A, I) \leqq n$, cf. [Hal, Sec. 1].) If no such n exists, we set $cd(A, I) = \infty$. However, if A is Noetherian of finite Krull dimension d, then $cd(A, I) \leqq d$; a well-known result due to Grothendieck [Gr1]. Another important vanishing result is that if I is defined set-theoretically by t equations, then $cd(A, I) \leqq t$; in particular $cd(A, I)$ provides a lower bound for the number of elements needed to define I set-theoretically.

The cohomological dimension has been studied by several authors; for example see Faltings [Fa1], Grothendieck [Gr1], Hartshorne [Ha1], [Ha2], Hartshorne and Speiser [H-S], Ogus [O], Peskine and Szpiro [P-S], and [S]. In this paper we will extend results from all of these articles. The main thrust of this paper is to give bounds on $cd(A, I)$, especially when I has small height (i.e., $\operatorname{Spec}(A/I)$ has small codimension) compared to the dimension of A.

Hartshorne [Ha1] proved an important result, widely known as the Hartshorne–Lichtenbaum vanishing theorem. This theorem basically states that if A is a complete local Noetherian domain, and I is an ideal of A then $\dim(A/I) \geqq 1$ iff $cd(A, I) < \dim(A)$. This theorem plays a crucial role in our proofs. In the same paper, Hartshorne proves a vanishing theorem for the cohomological dimension of the complement of a subvariety of projective space, which was later generalized by Ogus [O, Cor. 2.11] in equicharacteristic 0, and by Peskine and Szpiro [P-S, III, 5.5] in equicharacteristic $p > 0$. Combining their results gives:

Theorem 1.1. *Let (A, m) be a regular local ring of dimension d, and let I be an ideal of A. Assume that A is complete and has a separably closed residue field k which it*

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contains. Then the following are equivalent:

(*i*) $cd(A, I) \leqq d - 2$.
(*ii*) $\dim(A/I) \geqq 2$, *and* $\operatorname{Spec}(A/I) - \{m\}$ *is connected.*

Of course one can remove the assumption that A is complete with separably closed residue field if one assumes in (ii) that $\operatorname{Spec}(A/I) - \{m\}$ is "formally geometrically connected", which in essence reduces the problem to (ii) of Theorem 1.1.

Ogus proves this theorem in equicharacteristic 0 by using the theory of algebraic De Rham cohomology developed by Grothendieck and Hartshorne, while Peskine and Szpiro use the action of Frobenius in characteristic $p > 0$.

We are able (Theorem 2.9) to give a new proof of this result, independent of characteristic. The proof of Theorem 2.9 is based upon an induction theorem (2.5) which generalizes a result of Faltings [Fa1, Satz 1]. The need to have k separably closed in both Theorem A and in our later work necessitates some work, which discusses the splitting of primes in power series rings when the coefficient field is extended. Although we need this work in Sect. 2, we postpone the proofs until Sect. 4, since they are independent of the rest of this paper. Our main theorem in Sect. 4 is Theorem 4.2, which says that if a prime P in a formal power series ring with coefficients in a field k splits into t irreducible components in the power series ring over a field L containing k, then there is a finite separable extension K of k, contained in L, such that P already splits into t irreducible components in the power series ring with coefficients in K. This result generalizes some of the results of Chevalley's classic paper on power series rings [Ch].

The chief use of our induction theorem, however, is not to give a simpler and unified proof of Theorem 1.1, but is rather to give improved bounds for $cd(A, I)$ for arbitrary Noetherian rings A and ideals I of A.

Faltings [Fa1, Korollar 2] gave the most general vanishing result for ideals I in Noetherian rings A. Note that in general $cd(A, I) = \sup\{cd(A_P, I_P)\colon P \in \operatorname{Spec}(A)\}$, so there is no loss of generality in assuming A is local, which we will normally do throughout the paper. Let (A, m) be a local ring. In [Fa1] there is a bound for $cd(A, I)$ which in part depends upon the embedding dimension of A. Since this theorem, as well as many of the results in this paper are somewhat technical in nature (especially in the case that the base ring A is non-regular), we will restrict ourselves in this introduction to the case where A is a regular local ring.

Accordingly, let (A, m) be a regular local ring of dimension d, and let I be an ideal with $\operatorname{bight}(I) = b$. (Recall the *big height of* $I = \max\{ht(P)\colon P$ is a minimal prime of $I\}$.) Faltings shows that $cd(A, I) \leqq d - [(d-1)/b]$. In general this is the best possible result without additional assumptions (see [Ly1]). We are able to improve this result in several directions. We show in (5.1) that:

(0.1) If I is formally geometrically irreducible, $cd(A, I) \leqq d - 1 - [(d-2)/b]$, and this is best possible for all b, d, with $0 < b < d$.

and,

(0.2) If A/I is normal, $cd(A, I) \leqq d - [d/(b+1)] - [(d-1)/(b+1)]$.

These theorems are based on work in Sect. 3. We also give a bound on $cd(A, I)$ if one assumes that there is an integer t such that I_P is generated up to radical by t elements for all primes P with $\dim(A/P) \geqq 3$. For instance if one knows that I is locally generated up to radical by t equations on the punctured spectrum, then

these results are applicable. We refer the reader to Theorem 3.10 for a precise statement of this bound.

In Sect. 5 we specialize our work to regular local rings, give some examples, and try to extend Theorem 1.1 to the case where $cd(A, I) \leqq \dim(A) - 3$. Let (A, m) be a regular local ring containing a field. We are able to show under the assumption that $2 \operatorname{bight}(I) + 2 \leqq \dim(A)$, and provided no three minimal primes of I add to an m-primary ideal, then $cd(A, I) \leqq \dim(A) - 3$. (The best previously known result in this direction was that provided $3 \operatorname{bight}(I) < \dim(A)$, then $cd(A, I) \leqq \dim(A) - 3$.) This condition is not necessary; however we give necessary and sufficient conditions for $cd(A, I) \leqq \dim(A) - 3$ in the case where the number of minimal primes of I is at most 5 (under the condition that $2 \operatorname{bight}(I) + 2 \leqq \dim(A)$). This work generalizes in this case the theorem of Faltings, and other results in this paper.

Finally in Sect. 6 we apply our theorems to give vanishing results for the relative singular homology groups with complex coefficients of subvarieties of projective space. The fact that there is a relationship between the vanishing of local cohomology and the vanishing of relative singular homology was developed by Grothendieck and Hartshorne (see [Ha3]). For example, we show that if $Y^d \subseteq \mathbb{P}^n_{\mathbb{C}}$ is irreducible then $H_i(\mathbb{P}^n, Y; \mathbb{C}) = 0$ for $i \leqq [(n-1)/(n-\mathrm{d})]$. If Y is in addition normal, then $H_i(\mathbb{P}^n, Y; \mathbb{C}) = 0$ for $i \leqq [(n+1)/(n-d+1)] + [n/(n-d+1)] - 1$ (cf. Theorem 6.2). If Y is locally set-theoretically defined by $n - d$ equations, except on a Zariski closed set of dimension at most 1, then $H_i(\mathbb{P}^n, Y; \mathbb{C}) = 0$ for $i \leqq 2d - n + 1$, a generalization of the well-known result of Barth [Ba]. A little more work shows that in all of the above cases $\pi_i(\mathbb{P}^n, Y)$ are finite in the same range as $H_i(\mathbb{P}^n, Y; \mathbb{C}) = 0$.

2. An induction theorem

Let (A, m) be a local ring containing a field, and let I be an ideal of A. We denote the completion of A by $A^\wedge$, and by B we denote the ring obtained from $A^\wedge$ by separably closing its residue field and completing. Specifically, set $B = ((A^\wedge)^{sh})^\wedge$, the completion of the strict Henselization of $A^\wedge$. We note that B is a faithfully flat extension of A.

If M is any finitely generated A-module, set $N = M \otimes_A B$. Since B is faithfully flat over A, we have that $H^q_I(M) \otimes_A B \cong H^q_{IB}(N)$, and $H^q_I(M) = 0$ if and only if $H^q_{IB}(N) = 0$. Thus, in deciding whether $H^q_I(M)$ is zero, it suffices to consider these modules when the ring A is complete with separably closed residue field. In fact, it is necessary to incorporate this case in any assumptions made concerning the ideal I; e.g., certain vanishing results on the local cohomology of I are true if I remains prime after passing to B, but it is not enough to assume I is prime in A to obtain the same results.

Our first result is a sharpening of [Fa1, Satz1]. It is rather technical, but supplies us with the necessary inductive tool to prove our main results. The main improvements over the result of [Fa1] come from the use of the Hartshorne–Lichtenbaum local vanishing theorem [Ha1, Thm. 3.1] (which we refer to from now on as 'HLVT'). First, however, we need a couple of lemmas.

Definition 2.1. *Let A be a local ring. By* $\operatorname{mdim}(A)$ *we denote* $\min\{\dim(A/Q) | Q$ *is a minimal prime of* $A\}$. *If I is an ideal of A, we let* $c(I) = \operatorname{embdim}(A) - \operatorname{mdim}(A/IA)$.

Lemma 2.2. *Suppose that A is a universally catenary local ring containing a field, and*

I is an ideal of A. Then $\operatorname{mdim}(A) = \operatorname{mdim}(A^\wedge)$, *and* $c(I) = c(IA^\wedge)$. *If B is defined as above, then* $c(I) = c(IB)$.

Recall that a ring A is universally catenary if it is Noetherian and any A-algebra of finite type is catenary [Mat, p. 84]. Also if a local domain A is universally catenary, then $\hat{A}$ is equidimensional. [Ra]

Proof. Since $\operatorname{embdim}(A) = \operatorname{embdim}(A^\wedge)$, to prove that $c(I) = (IA^\wedge)$, it clearly suffices to prove the first statement of the lemma. (Note that A universally catenary implies that A/I is also.) Choose any minimal prime ideal P of A. Since A/P is a universally catenary domain, if follows that if Q is any minimal prime over $PA^\wedge$, then $\dim(A/P) = \dim(A^\wedge/PA^\wedge) = \dim(A^\wedge/Q)$. As minimal primes of $A^\wedge$ contract to minimal primes of A, it follows that $\operatorname{md} m(A^\wedge) = \operatorname{mdim}(A)$, as required. For the last statement it suffices to assume that $A = k[[x_1, \ldots, x_n]]$, $B = k^{\text{sep}}[[x_1, \ldots, x_n]]$, and I is a prime ideal of A. Then $c(I) = ht(I)$, and $c(IB) = \operatorname{bight}(IB) = ht(I)$ by [Ch, Prop. 9a]. □

Lemma 2.3. *Let A be a universally catenary local ring, and suppose that P is a prime ideal of A and I is an ideal of A. Then,*

i) $\dim(A/P) + \operatorname{mdim}(A_P) \geqq \operatorname{mdim}(A)$.
ii) $c(I) \geqq c(I_P)$, *if P contains I.*

Proof. Choose a prime ideal Q', minimal in A and contained in P, such that $\operatorname{mdim}(A_P) = \dim(A_P/Q'A_P)$. Since A is catenary, we obtain that $\dim(A/P) + \operatorname{mdim}(A_P) = \dim(A/Q') \geqq \operatorname{mdim}(A)$ which proves (i).

By [V, Thm. 2.2], $\operatorname{embdim}(A) - \operatorname{embdim}(A_P) \geqq \dim(A/P)$. Hence we obtain from i) that $c(I) = \operatorname{embdim}(A) - \operatorname{mdim}(A/I) \geqq \operatorname{embdim}(A_P) + \dim(A/P) - (\dim(A/P) + \operatorname{mdim}(A_P/I_P)) = c(I_P)$. □

Lemma 2.4. *Let A be a Noetherian local ring, and let I be an ideal of A. Suppose that B is a faithfully flat local extension of A with* $\dim(B) = \dim(A)$. *If* $H^q_{IA_P}(M_P) = 0$ *for an A-module M and for all $P \in \operatorname{Spec}(A)$ with* $\dim(A/P) > r$, *then* $H^q_{IB_Q}(M \otimes_A B_Q) = 0$ *for all $Q \in \operatorname{Spec}(B)$ with* $\dim(B/Q) > r$.

Proof. As $\dim(B) = \dim(A)$, we have that $\dim(B/mB) = 0$ [Mat, 21.A], where m is the maximal ideal of A. Let Q be a prime of B with $\dim(B/Q) > r$, and set $P = Q \cap A$. Since B/PB is faithfully flat over A/P, and B/mB is 0-dimensional, $\dim(A/P) = \dim(B/PB) \geqq \dim(B/Q) > r$. Therefore $H^q_{IA_P}(M_P) = 0$, and hence $H^q_{IB_Q}(M \otimes_A B_Q) = H^q_{IA_P}(M_P) \otimes_{A_P} B_Q = 0$. □

Theorem 2.5. *Let A be a Noetherian local ring containing a field, and let I be an ideal of A. Let $B = ((A^\wedge)^{sh})^\wedge$. Set $c = c(IB)$. Let M be any finitely generated A-module, and let $n > c$ be an integer. Assume that for all integers s, with $1 \leqq s \leqq c - 1$, and for all $q \geqq n - s$, the following hold:*

i) $H^q_{IA_P}(M_P) = 0$ *for all $P \in \operatorname{Spec}(A)$, such that $I \subseteq P$, and* $\dim(A/P) > s + 1$.
ii) $H^q_{IB_P}(M \otimes_A B_P) = 0$, *for all $P \in \operatorname{Spec}(B)$ such that $IB \subseteq P$,* $\dim(B/P) = s + 1$, *and $P + Q$ is primary to the maximal ideal of B for some minimal prime ideal Q of IB.*

Then $H^q_I(M) = 0$ *for all $q \geqq n$.*

Proof. We first reduce to the case where $A = B$ and $I = IB$. Assume we have shown the theorem in the case where A is already complete with separably closed residue

field, i.e., for $A = B$. The hypothesis i) of the theorem implies that $H^q_{IB_P}(M \otimes_A B_P) = 0$ for all primes $P \in \mathrm{Spec}(B)$ with $\dim(B/P) > s + 1$, by Lemma 2.4. Hence i) and ii) hold in B, and thus $H^q_{IB}(M \otimes_A B) = 0$ for all $q \geqq n$. As B is faithfully flat over A, we obtain that $H^q_I(M) = 0$ for all $q \geqq n$, as required. Thus we may assume that A is complete with separably closed residue field $k \subset A$.

Set $e = \mathrm{embdim}(A)$. Since A is a quotient of $k[[T_1, \ldots, T_e]]$, upon replacing A by this ring, and I by its preimage, we may assume that A is regular. In this case c is just the big height of I ($\mathrm{bight}(I)$), namely the largest height of any minimal prime of I. By descending induction on q we may assume that $H^q_I(M) = 0$ for all $q > n$, and we only need to show that $H^n_I(M) = 0$. We induct on the dimension of A/I, over regular local rings A, and ideals I. The starting point is when $\dim(A/I) = 0$. In this case $c = \dim(A)$, and $H^n_I(M) = 0$ for $n > \dim(A)$. We claim that this induction shows that the module $H^n_I(M)$ must have support only at m. If not, then localize at a prime P' minimal in the support of $H^n_I(M)$; we must show that the assumptions i) and ii) of the theorem are still satisfied for the ring $A_{P'}$ and for the ring $C = (((A_{P'})^\wedge)^{sh})^\wedge$. Observe that $c(I) \geqq c(IA_{P'})$ by (2.3 ii), and $c(IA_{P'}) = c(IC)$ by (2.2) (A is complete, hence universally catenary) Let Q be a prime ideal of C with $\dim(C/Q) = s + 1$. Set $Q' = Q \cap A$. Then $\dim(C/Q) \leqq \dim(A_{P'}/Q'A_{P'})$ (*as* $A_{P'} \to C$ is flat with closed fiber of dimension 0) $< \dim(A/Q')$. Hence $\dim(A/Q') > s + 1$, so that i) shows that $H^q_{IA_{Q'}}(M_{Q'}) = 0$ for all $q \geqq n - s$. It follows that $H^q_{IC_Q}(M \otimes_A C_Q) = 0$ for $q \geqq n - s$. By our choice of Q, it follows that condition ii) is satisfied. Let $Q_{P'} \in \mathrm{Spec}(A_{P'})$. If $\dim(A_{P'}/Q_{P'}) > s + 1$, then $\dim(A/Q) > s + 1$, and $H^q_{IA_Q}(M_Q) = 0$ for $q \geqq n - s$. Hence i) is also satisfied, which by induction shows that $H^q_{IA_{P'}}(M_{P'}) = 0$ for $q \geqq n$, contradicting the choice of P'. This proves our claim.

Let $D = (A/I) \hat{\otimes}_k A$ be the complete tensor product. The natural surjection of A onto A/I induces an (A/I)-linear surjection of D onto A/I whose kernel Δ is generated by the e elements $T_i \hat{\otimes} 1 - 1 \hat{\otimes} T_i$, where $A = k[[T_1, \ldots, T_e]]$. Set $N = D \otimes_A M = (A/I) \hat{\otimes}_k M$.

Since $ID = (A/I) \hat{\otimes}_k I \subset \Delta$, there is a spectral sequence for the composition of functors, $E_2^{p,q} = H^p_\Delta(H^q_{ID}(N)) \Rightarrow H^{p+q}_\Delta(N)$. Because D is a flat A-algebra, we have that $H^q_{ID}(N) \cong D \otimes_A H^q_I(M)$.

Set $d = \dim(A/I)$, and consider the term,

$$E_2^{d,n} = H^d_\Delta(D \otimes_A H^n_I(M)) \Rightarrow E_\infty^{d,n}(N)\,.$$

Since $d + n > d + c \geqq d + (e - d) = e$, and since Δ is generated by e elements, $H^{d+n}_\Delta(N) = 0$, so $E_\infty^{d,n}(N) = 0$.

All incoming differentials to $E_2^{d,n}$ are zero, since they come from $E_r^{d-r,n+r-1}$ for $r > 1$, and $H^{d-r}_\Delta(D \otimes_A H^{n+r-1}_I(M)) = 0$, as by assumption $H^q_I(M) = 0$ for $q > n$.

The outgoing differentials land in $E_r^{d+r,n-(r-1)}$, and we claim that $H^{d+r}_\Delta(D \otimes_A H^{n-(r-1)}(M)) = 0$ for $r \geqq 2$. Indeed, if $r > c$, then $d + r > d + c \geqq \dim(A) = e$, and since Δ is generated by e elements, H^{d+r}_Δ (any module) $= 0$. Suppose that $r \leqq c$, and set $s = r - 1$. In general $H^{n-s}_I(M)$ is the direct limit of modules L_α, where $\{L_\alpha\}$ is the collection of finitely generated submodules of $H^{n-s}_I(M)$. Hence $H^{d+r}_\Delta(D \otimes_A H^{n-s}_I(M)) = \varinjlim(H^{d+r}_\Delta(D \otimes_A L_\alpha)) = \varinjlim(H^{d+r}_\Delta((A/I) \hat{\otimes}_k L_\alpha))$, since direct limits commute with the tensor product and are exact [R].

Let $P_1, \ldots, P_u$ be the minimal primes of the support of L_α in A for a fixed α, and let $Q_1, \ldots, Q_v$ be the minimal primes of A/I. We claim that the minimal primes of $(A/I) \hat{\otimes}_k L_\alpha$ are $\Omega_{ji} = \mathrm{nilrad}(Q_j \hat{\otimes}_k A + (A/I) \hat{\otimes}_k P_i) \subset D$. To prove this,

we first observe that by Corollary 4.5 the ideals Ω_{ji} are prime, since k is separably closed and is the residue field of A and A/I. Note that A and A/I embed into $D = (A/I)\hat{\otimes}_k A$. We have that $\Omega_{ji}(D/P_iD) = \text{nilrad}(Q_j(D/P_iD))$ is a prime ideal, and its contraction to A/I is Q_j since the map from A/I to $D/P_iD = (A/I)\hat{\otimes}_k(A/P_i)$ is faithfully flat. It follows that the ideals $\text{nilrad}(Q_j(D/P_iD))$ are exactly the minimal prime ideals of D/P_iD, since any minimal prime of D/P_iD must contract to one of the Q_j (by flatness). As D is flat over A we see that $\text{Ass}(D\otimes_A L_\alpha) = \bigcup_{P\in \text{Ass}(L_\alpha)} \text{Ass}(D/PD)$ by [Mat, 21.B]. Any minimal prime of $D\otimes_A L_\alpha$ is an associated prime of this module, and therefore is associated to D/PD for some $P\in\text{Ass}(L_\alpha)$. In particular, if N is a minimal prime of $D\otimes_A L_\alpha$, then N contains P_iD for some i. By our calculation above, this implies that N contains Ω_{ji} for some i. However, all of the Ω_{ji} are associated to $D\otimes_A L_\alpha$ by the formula above. Hence N must be equal to one of the Ω_{ji}, which proves our claim.

We know that $\dim(A/Q_j) \leqq d = \dim(A/I)$, and by i) $\dim(A/P_i) \leqq s+1 = r$, and hence $\dim(D/\Omega_{ji}) \leqq d+r$. Suppose that $\dim(A/P_i) < r$ for every i. Then $\dim((A/I)\hat{\otimes}_k L_\alpha) < d+r$, and so $H^{d+r}_{\Delta}((A/I)\hat{\otimes}_k L_\alpha) = 0$, as we claimed. Thus we may assume that $\dim(A/P_i) = r$ for some i. Then ii) guarantees that $P_i + Q_j$ is not m-primary for every j. Hence in D, the ideal $\Delta + \Omega_{ji}$ is not η-primary for any j, where η is the maximal ideal of D. Thus $\Delta + \Omega_{ji}$ is not η-primary for any minimal prime Ω_{ji} of D having dimension equal to $d+r = \dim(C)$, where $C = (A/I)\hat{\otimes}_k(A/\text{ann}(L_\alpha))$. Now apply HLVT [Ha1, Thm. 3.1]; C is complete, so we may conclude that $H^{d+r}_{\Delta}(M) = 0$ for any module M. In particular, $H^{d+r}_{\Delta}((A/I)\hat{\otimes}_k L_\alpha) = 0$ in this case. Thus in any case the outgoing differentials are zero as well, and we may conclude that $H^d_{\Delta}(D\otimes_A H^n_I(M)) = 0$. Since $H^n_I(M)$ is supported only at the maximal ideal, it is the direct limit of modules L having finite length. For any such L, we have that $H^d_{\Delta}(D\otimes_A L) \cong H^d_{\Delta}((A/I)\hat{\otimes}_k L) \cong H^d_{\Delta}((A/I)\otimes_k L)$. The support of $(A/I)\otimes_k L$ can be identified with $\text{Spec}(A/I)$. The intersection of Δ with this support is the closed point $\{m/I\}$, where m/I is the maximal ideal of A/I. Hence, $H^d_{\Delta}((A/I)\otimes_k L) = H^d_{m/I}((A/I)\otimes_k L) \cong H^d_{m/I}(A/I)\otimes_k L$, where the latter isomorphism follows as L is flat over k. Passing to the direct limit, we obtain that $H^d_{\Delta}(D\otimes_A H^n_I(M)) \cong H^d_{m/I}(A/I)\otimes_k H^n_I(M) = 0$. However, since $H^d_{m/I}(A/I) \neq 0$ [Gr1, 6.4], and the tensor product is over a field, we conclude that $H^n_I(M) = 0$, as required. □

Remark 2.6. The hypotheses of Theorm 2.5 are fulfilled if for all s between 1 and $c-1$ and for all $q \geqq n-s$, the module $H^q_{IA_P}(M_P) = 0$ for all primes P with $\dim(A/P) = s+1$. This is Faltings' original theorem [Fa1, Satz 1].

Remark 2.7. Assume that $\text{nilrad}(IB)$ is prime, in the context of Theorem 2.5. Then no prime $P \supseteq IB$ such that $\dim(B/P) = s+1 (1 \leqq s \leqq c-1)$, and such that $H^q_{IB_P}(M\otimes_A B_P) \neq 0$, satisfies the condition that $\text{nilrad}(IB) + P$ is primary for the maximal ideal of B. Hence condition ii) of Theorem 2.5 is vacuous in this case, and so the hypotheses of Theorem 2.5 will be satisfied if $H^q_{IA_P}(M_P) = 0$ for all primes P with $\dim(A/P) = s+2$, and for all $q \geqq n-s$. This observation will allow us to improve Faltings' estimate in case I is formally geometrically irreducible (i.e., $\text{nilrad}(IB)$ is prime).

Remark 2.8. If $c \leqq 1$ there are no integers s with $1 \leqq s \leqq c-1$. Hence (i) and (ii) are vacously satisfied. Hence $H^q_I(M) = 0$ for $q > c$. This can also be seen directly. Indeed if $c = 0$ then A is regular and $I = 0$. If $c = 1$ then $A^\wedge$ is the surjective image

of a regular ring A' of dimension $e = \operatorname{embdim}(A)$, and the preimage I' of $I\hat{A}$ in A' has bigheight 1. Hence $\sqrt{I'}$ is principal and the result follows.

To finish this section we wish to illustrate how Theorem 2.5 allows a new characteristic-free proof of theorems of Hartshorne and Ogus in characteristic 0, and Peskine and Szpiro in characteristic p [P-S]. These theorems give necessary and sufficient conditions for an ideal I in a complete regular local ring of dimension d with separably closed residue field to satisfy $H_I^{d-1}(M) = 0$ for all modules M.

Theorem 2.9. *Let (A, m) be a regular local ring of dimension d containing a field, and let I be an ideal of A. Then $H_I^q(M) = 0$ for all $q \geqq d - 1$ if and only if*

i) $\dim(A/I) \geqq 2$, *and*
ii) $\operatorname{Spec}(A/I) - \{m\}$ *is formally geometrically connected, i.e., If* $B = ((A\hat{\ })^{sh})\hat{\ }$, then $\operatorname{Spec}(B/IB) - \{mB\}$ *is connected.*

Proof. By the remarks in the beginning of this section, we may assume that $A = B = k[[x_1, \ldots, x_d]]$, with k separably closed. First we will show the necessity of conditions i) and ii). If I is m-primary, then $H_I^d(A) \neq 0$ [Gr1, 6.14]. If I has a minimal prime over-ideal P of dimension 1, then $H_{I_P}^{d-1}(A_P) \neq 0$, and therefore $H_I^{d-1}(A) \neq 0$. Thus $\dim(A/I) \geqq 2$. Assume that $\operatorname{Spec}(A/I) - \{m\}$ is disconnected. Then there are two ideals J and K such that $J \cap K$ has the same radical as I, both J and K contain I but do not have the same radical, and $J + K$ is m-primary. By HLVT, $H_J^d(M) = H_K^d(M) = 0$ for any module M. The Mayer–Vietoris sequence of local cohomology, applied to J and K, shows that $H_I^{d-1}(A)$ maps onto $H_{J+K}^d(A) \neq 0$. Thus condition ii) is also necessary.

For the sufficiency of conditions i) and ii), we first consider the case where I is prime. Set $n = d - 1$. By Theorem 2.5 and Remark 2.7, to see that $H_I^q(M) = 0$ for all $q \geqq n$, it suffices to prove that $H_{IA_P}^q(M_P) = 0$ for all $q \geqq n - s$, and all P with $\dim(A/P) = s + 2$, for $1 \leqq s \leqq \operatorname{bight}(I) - 1 = ht(I) - 1$, since I is prime. However, $\dim(A/P) = s + 2$ implies that $\dim(A_P) = d - s - 2 = n - s - 1$. Hence if $q \geqq n - s$, then $q > \dim(A_P)$, so that $H_{IA_P}^q(M_P) = 0$.

Now consider the general case. Let $P_1, \ldots, P_r$ be the minimal primes over I, and use induction on r to prove the theorem, the case $r = 1$ being done above. Every P_i must have dimension at least 2; if not, say P_1 has dimension 1. Then $P_1 + \bigcap_{i \neq 1} P_i$ is m-primary, and this gives a disconnection of the punctured spectrum of A/I, in contradiction to assumption ii). Consider the graph Γ with r vertices $1, \ldots, r$, where i and j are joined by an edge if and only if $P_i + P_j$ is not m-primary. Assumption ii) shows that Γ is connected. Any finite connected graph has a spanning tree [B, Cor. 5, p. 7]. The number of edges in a tree is smaller than the number of vertices, so there is a vertex from which only one edge emanates [B, Cor. 7, p. 8]. The original graph minus this vertex is connected. Hence we may assume that the graph $\Gamma - \{1\}$ is connected, and set $J = \bigcap_{i \neq 1} P_i$. The ideal J satisfies the assumptions of the theorem, so by induction, $H_J^q(M) = 0$ for all M, and for all $q \geqq d - 1$. Also $H_{P_1}^q(M) = 0$ for all $q \geqq d - 1$. The Mayer–Vietoris sequence applied to J and P_1 yields an exact sequence, $H_J^{d-1}(M) \oplus H_{P_1}^{d-1}(M) \to H_I^{d-1}(M) \to H_{J+P_1}^d(M)$. Since $J + P_1$ is not m-primary, $H_{J+P_1}^d(M) = 0$ by HLVT. Hence $H_I^{d-1}(M) = 0$. By HLVT, $H_I^d(M) = 0$, while $H_I^q(M) = 0$ for $q \geqq d + 1$ as $\dim(A) = d$. □

3. Vanishing theorems

In this section we prove our main vanishing results for local cohomology. They are fairly easy inductive consequences of our Theorem 2.5. Let (A, m) be a local noetherian ring containing a field, let I be an ideal of A with $c = c(IA^\wedge) > 0$, and M be a finitely generated A-module of dimension d. G. Faltings [Fa1, Korollar 2] proved that

$$H_I^q(M) = 0 \qquad \text{for } q > d - [(d-1)/c] . \tag{3.1}$$

Faltings also proved (cf. [Fa1, Korollar 4]) that if I_P is generated set-theoretically by r elements for every prime $P \supseteq I$ with $\dim(A/P) \geqq 2$, then

$$H_I^q(M) = 0 \qquad \text{for } q > c + r - 1 . \tag{3.2}$$

It is known that without further restrictions, (3.1) is best possible [Ly1]. In this section we sharpen these results by putting additional assumptions on the ideal I. If I is formally geometrically irreducible (see Definition 3.6 below), we are able to show (cf. Theorem 3.8) that provided $0 < c(IA^\wedge) < \dim(A)$ then with $d = \dim(M)$,

$$H_I^q(M) = 0 \qquad \text{for } q > d - 1 - [(d-2)/c] . \tag{3.3}$$

In Sect. 5 we will show that (3.3) is best possible. In Theorem 3.9 we generalize the vanishing theorems further by assuming more about I. Let A be an excellent local ring containing a field, let I be formally geometrically irreducible with $c(IA^\wedge) = c$, and assume that $(A/I)_P$ is normal for all primes P containing I with $\dim(A/P) \geqq 3$. If M is a finitely generated A-module of dimension d then,

$$H_I^q(M) = 0 \qquad \text{for } q > d - [d/(c+1)] - [(d-1)/(c+1)] . \tag{3.4}$$

We are also able to generalize Faltings' result (3.2). We show in Theorem 3.10 below that if A is a local ring containing a field and I is a formally geometrically irreducible ideal such that I_P is set-theoretically generated by r elements for all primes P containing I with $\dim(A/P) \geqq 3$, then

$$H_I^q(M) = 0 \qquad \text{for all } q > c + r - 1, \quad \text{for all } A\text{-modules } M . \tag{3.5}$$

We begin with a definition.

Definition 3.6. We say that an ideal I in a local ring A is formally geometrically irreducible if B/IB has a unique minimal prime, where $B = ((A^\wedge)^{sh})^\wedge$), the completion of the strict Henselization of the completion of A.

Our main technical tool of this section is (3.7) below which is a consequence of our induction theorem (2.5).

Lemma 3.7. *Let A be a local noetherian ring containing a field and let I be a formally geometrically irreducible ideal of A. Let $n > c = c(IA^\wedge)$ be an integer. Let M be any finitely generated A-module. Assume for all integers s, $1 \leqq s \leqq c-1$, and all primes $P \supseteq I$ with $\dim(A/P) = s + 2$ that the local cohomology modules $H^q_{IA_P}(M_P) = 0$ for all $q \geqq n - s$. Then $H_I^q(M) = 0$ for all $q \geqq n$.*

Proof. Set $B = ((A^\wedge)^{sh})^\wedge$). By assumption nilrad$(IB)$ is prime. Hence there does not exist any prime P in Spec(B) such that P contains IB, $\dim(B/P) > 0$, and $P + Q$ is primary to the maximal ideal of B for some minimal prime Q of IB. This gives that condition (ii) of Theorem 2.5 is vacuously satisfied. Hence all of the hypotheses of (2.5) are satisfied and the conclusion of (2.5) gives the result. □

Theorem 3.8. *Let A be a local ring containing a field, let I be a formally geometrically irreducible ideal in A with $0 < c(IA^\wedge) = c < d$, where d is the dimension of a finitely generated A-module M. Then $H_I^q(M) = 0$ for $q > d - 1 - [(d-2)/c]$. In particular if $0 < c < \dim(A)$, then $cd(A, I) \leqq \dim(A) - 1 - [(\dim(A) - 2)/c]$.*

Proof. Set $B = ((A^\wedge)^{sh})^\wedge$. Since $A^\wedge$ is universally catenary, (2.2) shows that $c = c(IB)$, and the remarks at the beginning of Sect. 2 then show that we may assume that $A = B$. If $c = 1$, then $\sqrt{IB}$ is principal (see (2.8)), and the result is trivial. Henceforth we assume that $c \geqq 2$. Set $n(t) = t - [(t-2)/c]$. Then n is a nondecreasing function of t. In particular for $t = d$, using that $d \geqq c + 1$, we see that $n = n(d) > c$. We apply (3.7) with this n. Choose any s, $1 \leqq s \leqq c - 1$, and let P be a prime containing I with $\dim(A/P) = s + 2$. A is complete, hence universally catenary, so (2.3) shows that $c(I_P) \leqq c$. As $\dim(M_P) \leqq d - s - 2$, (3.1) implies that $H_{I_P}^q(M_P) = 0$ for $q \geqq d - s - 1 - [(d - s - 3)/c]$. To apply (3.7) it suffices to show that $n - s \geqq d - s - 1 - [(d - s - 3)/c]$, which reduces to $1 \geqq [(d-2)/c] - [(d - s - 3)/c]$. As $s + 1 \leqq c$ the inequality follows. We can now apply (3.7) to obtain the first conclusion of the theorem. The last statement follows from setting $M = A$ and using [Ha1, Sec. 1]. □

Theorem 3.9. *Let A be an excellent local ring containing a field. Let I be a formally geometrically irreducible ideal of A with $0 < c = c(IA^\wedge)$, and assume that $(A/I)_P$ is normal for all primes $P \supseteq I$ with $\dim(A/P) \geqq 3$. Let M be a finitely generated A-module of dimension d. Then $H_I^q(M) = 0$ for $q \geqq d + 1 - [d/(c+1)] - [(d-1)/(c+1)]$. In particular if $n = \dim(A)$ and $0 < c$ then*

$$cd(A, I) \leqq n - [n/(c+1)] - [(n-1)/(c+1)] .$$

Proof. We induct upon d to prove the theorem. If $d \leqq c + 1$, the result follows from (3.1). Henceforth we assume that $d > c + 1$, and the result holds for modules of dimension $< d$. If $c = 1$, the conclusion holds (by (2.8)). Thus we also assume that $c \geqq 2$. We again apply (3.7). Set $n = d + 1 - [d/(c+1)] - [(d-1)/(c+1)]$. To apply (3.7) we first show that $n > c$. This reduces to proving that $d + 1 > (2d - 1)/(c+1) + c$. Clearing the denominator yields the equivalent inequality $c^2 - cd + d - 2 < 0$, i.e. that $(c-1)(c+1-d) < 1$ which holds by our constraints upon c. Thus $n > c$.

Let $1 \leqq s \leqq c - 1$, and let $P \supseteq I$ be a prime with $\dim(A/P) = s + 2$. By (2.2) and (2.3) $c(I(A_P)^\wedge) = c(I_P) \leqq c$. The dimension of $M_P \leqq d - s - 2$. As $\dim(A/P) \geqq 3$, A/P is normal. Since A is excellent $(A_P/I_P)^\wedge$ is also normal and (4.4) shows that I_P is formally geometrically irreducible. We may apply the induction to A_P, I_P, and M_P to obtain that $H_{IA_P}^q(M_P) = 0$ for $q \geqq (d - s - 2) + 1 - [(d - s - 2)/(c+1)] - [(d - s - 3)/(c+1)]$. Now (3.7) will finish the proof if we show that $n - s \geqq (d - s - 2) + 1 - [(d - s - 2)/(c+1)] - [(d - s - 3)/(c+1)]$. This reduces to $2 \geqq (([d/(c+1)] - [(d - s - 2)/(c+1)]) + ([(d-1)/(c+1)] - [(d - s - 3)/(c+1)]))$ which is true since $s + 2 \leqq c + 1$. The last statement of the proof follows by setting $M = A$. □

Finally we give our last vanishing result.

Theorem 3.10. *Let A be a local ring containing a field, let I be a formally geometrically irreducible ideal of A with $c = c(IA^\wedge)$, and assume that I_P is set-theoretically generated by r elements for all primes $P \supseteq I$ with $\dim(A/P) \geqq 3$. Then $H_I^q(M) = 0$ for all $q > c + r - 1$, and for all A-modules M, i.e. $cd(A, I) \leqq c + r - 1$.*

Proof. Set $n = r + c$, and apply (3.7). For the last statement set $M = A$. □

Remark 3.11. Let A be a universally catenary local ring containing a field, let I be an ideal of A with $c = c(IA^{\wedge})$, and let M be a finitely generated A-module of dimension d. Set $c(v) = \max\{c(I_P)\colon P \supseteq I, \dim(A/P) \geqq v\}$. By (2.3) the function $c(v)$ is non-increasing. All of the results of this section were proven assuming only that $c(0)$ is known. However, if more information of $c(v)$ is available, stronger results are possible. For example:

(i) Assume that there is a positive integer $t \leqq c = c(0)$ such that $c(t+1) \leqq t$. Then $H^q_I(M) = 0$ for $q > d - 1 - [(d-1-c)/t]$.

(ii) Assume that I is formally geometrically irreducible and there is a positive integer $t \leqq c = c(0)$ such that $c(t+2) \leqq t$. Then $H^q_I(M) = 0$ for $q > d - 2 - [(d-c-2)/t]$.

(iii) Assume that A is excellent, I is formally geometrically irreducible, A_P/I_P is normal for every prime $P \supseteq I$ with $\dim(A/P) \geqq 3$, and there exists a positive integer $t \leqq c$ such that $c(t+2) \leqq t$. Then $H^q_I(M) = 0$ for $q - 2 - [(d-c-1)/(t+1)] - [(d-c-2)/(t+1)]$.

We omit the proofs since no new ideas are involved. Of course, if precise values of $c(v)$ are known, even sharper estimates are possible. Also, if in addition to information about $c(v)$ it is known that I_P is set-theoretically generated by r elements for primes $P \supseteq I$ with $\dim(A/P) \geqq m$, where r and m are some fixed integers, even sharper results are possible, but the formulas are quite cumbersome. We conclude this section with several examples to illustrate the strengths of the various estimates.

Example 3.12. Let k be a field of characteristic 0, and let $X = (x_{ij})$ be a generic 3×6 matrix. Consider the ideal I generated by the 3×3 minors of X in the ring $B = k[x_{ij}]$. Let $A = B_{(x_{ij})}$. Then $\dim(A) = 18$, $\mu(I) = 20$, and $ht(I) = c(I) = 5$. Furthermore (e.g. see [B-V]), A/I is normal, Cohen–Macaulay, and if $P \neq (x_{ij})$, $\mu(I_P) \leqq 10$. We apply our various estimates to compute $cd(A, I)$. HLVT gives that $cd(A, I) \leqq 19$. Faltings estimate (3.1) gives that $cd(A, I) \leqq 18 - [17/5] = 15$. Since I is homogeneous it is formally geometrically irreducible so that our Theorem 3.8 gives $cd(A, I) \leqq 18 - 1 - [16/5] = 14$. Since A/I is normal and I is formally geometrically irreducible we may also apply Theorem 3.9 to conclude that $cd(A, I) \leqq 18 - [18/6] - [17/6] = 13$. Finally we can apply (3.10) to obtain that $cd(A, I) \leqq 5 + 10 - 1 = 14$. Obviously (3.9) gives the best estimate.

The actual cohomological dimension of I, however, is 10, by the remark after this example. This can be obtained by our methods, or even Faltings' original theorem by using induction upon the size of the matrix. Clearly, though, the theorems above are not always the best possible—this is because information concerning localizations of an ideal will not in general tell if I is defined up to radical by a small number of equations.

Remark 3.13. Let k be characteristic 0 and let I be generated by the $r \times r$ minors of an $r \times s$ generic matrix X $(r \leqq s)$. Let $B = k[x_{ij}]$, and $A = B_{(x_{ij})}$. Then $cd(A, I) = cd(B, I) = r(s-r) + 1$.

Proof. We sketch the proof, which was probably known. If I is ideal in a polynomial ring generated by homogeneous equations of the same degree, then the analytic spread of I is the dimension of $k[\mathrm{I}]$ as a subring of the polynomial ring. In the case of this example, $k[I]$ is just the homogeneous coordinate ring of the

Grassman variety of r-planes in s-space, which is known to have dimension $r(s-r)+1$. Therefore, I is generated up to radical by $r(s-r)+1$ equations, so that $cd(A, I) \leqq r(s-r)+1$.

On the other hand, as Hochster pointed out to us, the map from $C = k[I]$ to $B = k[x_{ij}]$ splits as a C-module since C is the ring of invariants of a linearly reductive group acting linearly on B. Hence the local cohomology $H^q_{IC}(C)$ injects into $H^q_{IC}(B)$ for all q. In particular, $H^q_I(B)$ cannot be zero for $q = \dim(C) = r(s-r)+1$, and consequently $H^q_I(A) \neq 0$. Thus $r(s-r)+1 \leqq cd(A, I)$, which proves the remark. □

Suppose that we assume in this example that $\operatorname{char} k = p > 0$. Since A/I is Cohen–Macaulay, [P-S] shows that $cd(A, I) = ht(I) = 5$.

Example 3.14. To obtain a better idea of the various estimates, let us consider a complete regular local ring (A, m) of dimension 60 containing its separably closed residue field, and let $I \subseteq A$ be an ideal with $\operatorname{bight}(I) = 5$. HLVT shows that $cd(A, I) \leqq 59$, while Faltings' result (3.1) shows that $cd(A, I) \leqq 60 - [59/5] = 49$. If I is formally geometrically irreducible then (3.8) may be applied to obtain that $cd(A, I) \leqq 60 - [58/5] - 2 = 48$. If in addition $(A/I)_P$ is normal for all $P \supseteq I$ with $\dim A/P \geqq 3$, then we can use (3.9) to get $cd(A, I) \leqq 60 - [60/6] - [59/6] = 41$. Finally suppose that I_P is set-theoretically generated by r elements for all primes $P \supseteq I$ with $\dim(A/P) \geqq 3$, and I is formally geometrically irreducible. Then (3.10) applies to show that $cd(A, I) \leqq r + 4$, which is < 41 if $r < 37$.

Remark 3.15. If we judge the efficacy of our estimates by the size of $\dim(A) - cd(A, I)$, then our results show that when I is formally geometrically irreducible, we can do one better than in general, while if A/I is normal, we can do approximately twice as well. These results suggest that if one assumes that A/I satisfies Serre's properties S_i and R_{i-1}, then better estimates can be obtained as i increases. However, as far as we can tell, our proofs can not be extended to prove any such result.

Remark 3.16. This example shows that, at least in some cases, the bound given by (3.9) is best possible. Let $X = (x_{ij})$ be a $2 \times n$ matrix of indeterminates. Let A be the polynomial ring in the $2n$ variables x_{ij} localized at the irrelevant ideal. Set I equal to the ideal generated by the 2×2 minors of X. Then A/I is normal. In 3.13 we showed that $cd(A, I) = 2(n-2)+1 = 2n-3$. We get from (3.9) that $cd(A, I) \leqq 2n - 2 - [(2n-(n-1)-1)/n] - [(2n-(n-1)-2)/n] = 2n-3$. Hence the estimate is exact in this case. However, we have not in general found examples for all d and c showing that (3.9) is best possible.

4. Prime ideals in power series rings

In this section we prove several results about the extension of prime ideals in power series rings which were needed in Sect. 2. These results, particularly Corollary 4.4, were crucial to some of our theorems. In particular, Corollary 4.4 is needed to prove the vanishing theorems for normal ideals in characteristic $p > 0$. The main result is Theorem 4.2 which extends results in Chevalley's paper [Ch], and Gilmer's paper [G] (see also [Gr2, IV, 7.5]) and is of independent interest. It says that if a prime in a power series ring splits into several components after a field extension

of the coefficients, then it must already split in some finite separable subextension. Our first lemma shows this for polynomial rings—a result which is probably well-known, but for which we could find no reference.

Lemma 4.1. *Let P be a prime in $R = k[x_1, \ldots, x_n]$, where k is a field and R is a polynomial ring over k. Let L be an extension field of k, and set $S = L[x_1, \ldots, x_n]$. If $\{Q_1, \ldots, Q_t\}$ are distinct minimal primes over PS, then there is a finite separable extension W of k, $W \subseteq L$, such that $PW[x_1, \ldots, x_n]$ has t distinct minimal primes over it, namely $Q_i \cap W[x_1, \ldots, x_n]$, $1 \leqq i \leqq t$.*

Proof. Let F be the separable closure of k in L. First we show that PR_F has t distinct minimal primes over it in R_F, where we write $R_G = G[x_1, \ldots, x_d]$ for any extension ring G of k. It is enough to see that any prime Q in R_F cannot have more than 1 minimal prime when extended to R_L. Let N be the quotient field of R_F/Q. Then $R_L/QR_L = (R_F/Q) \otimes_F L \subset N \otimes_F L$, so it suffices to see that every zero-divisor in $N \otimes_F L$ is nilpotent. As F is separably algebraically closed in L, this follows at once from [Ja, Thm. 24, pg. 202].

Thus we may assume that L is separable algebraic to prove the lemma. In this case choose a generating set of each Q_i. Adjoin all the coefficients of these polynomials to F, and call that field W. Note that W is a finite separable extension of F, and PR_W must have t distinct minimal primes, namely $Q_i' = Q_i \cap R_W$. For Q_i' clearly is a prime containing PR_W, and $\dim(R_W/Q_i') = \dim(R_L/Q_i)$ (by integrality) $= \dim(R/P)$, since $Q_i \cap R = P$. Hence Q_i' is minimal over PR_W, and they are necessarily distinct by construction. □

We now will prove the same result for power series rings. First we establish some notation. Let $R = k[[x_1, \ldots, x_n]]$, where k is a field, and R is a power series ring over k. If G is any extension ring of k, let R_G denote the ring $G[[x_1, \ldots, x_n]]$. Let P be a prime in R, and by S denote R/P. By S_G denote R_G/PR_G, if G is an extension of k. If a ring C is a domain, by $F(C)$ we will denote the quotient field of C.

Theorem 4.2. *Adopt the notation of the above paragraph, and assume that L is a field extension of k. If PR_L has exactly t distinct minimal primes $Q_1, \ldots, Q_t$ lying over it in R_L, then there is a finite separable extension K of k, $K \subseteq L$, such that PR_K has exactly t distinct minimal primes lying over it in R_K, which are the contractions of the Q_i. In particular, there is a one-to-one correspondence between the minimal primes of PR_L and PR_K.*

Proof. Set $d = \dim(S)$. After a change of variables, if necessary, we may assume that $A = k[[x_1, \ldots, x_d]] \subset S$, and S is a finite A-module. Let $P' = P \cap A[x_{d+1}, \ldots, x_n]$. Note that $A[x_{d+1}, \ldots, x_n]/P'$ maps onto S, has the same dimension, and is a domain; thus $A[x_{d+1}, \ldots, x_n]/P' = S$. Observe also that $P' \cap A = P \cap A = (0)$.

Let $B = A_L = L[[x_1, \ldots, x_d]]$. We claim that $Q_i \cap B = (0)$ for all i. Suppose not, set $q_i = Q_i \cap B$, with $q_i \neq (0)$. We have that $R_L/PR_L = R_L/P'R_L = B[x_{d+1}, \ldots, x_n]/P'B[x_{d+1}, \ldots, x_n]$ is integral over B, by the fact that the x_i, with $d + 1 \leqq i \leqq n$, satisfy integral equations over A. Hence R_L/Q_i is integral over $B/Q_i \cap B$, which in turn implies that $\dim(R_L/Q_i) = \dim(B/q_i) < \dim(B) = d$. However, [Ch, Prop. 9] shows that $\dim(R_L/Q_i) = \dim(R/P) = d$. This contradiction shows that $Q_i \cap B = (0)$.

As $Q_i \cap B = (0)$ and $P' \cap A = (0)$, we must have that there are t minimal primes

over $P'F(B)[x_{d+1}, \ldots, x_n]$, namely $Q_i \cap F(B)[x_{d+1}, \ldots, x_n]$. By Lemma 4.1, this implies that the same is true in a finite separable extension W of $F(A)$ contained in $F(B)$. Consider the integral closure of A in W; call this C. Choose a primitive element $\gamma \in F(B)$ which generates W over $F(A)$. We can write γ as a quotient f/g of two elements of C, and from elementary commutative algebra, we may assume that $g \in A$. Since f is integral over A, we see that $f \in B$, as B is integrally closed and $f \in F(B)$. Thus, f satisfies a separable integral equation over A, and $W = F(A)(f)$.

By [G, Thm. 4.1], the field $K' = k(\text{coefficients of } f) \subseteq L$ has the property that $[K' : k]_{\text{sep}} < \infty$, and has finite exponent e over k. Let $\beta = f^q$, where $q = p^e$. Since W is separable over $F(A)$, we have that $W = F(A)(\beta)$. Let K be the field $k(\text{coefficients of } \beta)$. Then K is finite separable over k, and $W \subset F(A_K)$. (Note that this shows that $f \in F(A_K)$, which in turn means that the coefficients of f are in K; thus $K' = K$ and K' was actually separable. However, we shall not need this comment.)

We claim that PR_K has t distinct minimal primes over itself. It suffices to see that $Q_i \cap R_K$ are distinct (by flatness they are necessarily minimal over PR_K), and for this it suffices to see that $Q_i \cap F(A_K)[x_{d+1}, \ldots, x_n]$ are distinct. However, $F(A_K) = K((x_1, \ldots, x_d))$ contains W, and by the above argument, $Q_i \cap W[x_{d+1}, \ldots, x_n]$ are distinct. Again by faithful flatness, every minimal prime of PR_K is the contraction of one of the Q_i, which finishes the proof. □

Corollary 4.3. *Adopt the notation of Theorem* 4.2. *Suppose that k is separably closed in L. Then* $\text{nilrad}(PR_L)$ *is a prime ideal.*

Proof. As k is separably closed in L, there is only one minimal prime above PR_L, by Theorem 4.2. This gives the statement of the Corollary. □

Corollary 4.4. *Adopt the notation of Theorem* 4.2. *Assume that R/P is normal, and that L is any field extension of k. Then* $\text{nilrad}(PR_L)$ *is a prime ideal.*

Proof. By Theorem 4.2 it suffices to prove this corollary when L is a finite separable extension field of k. Set $S = R/P$. We claim that k is algebraically closed in $F(S)$. If not, let K be the algebraic closure of k in $F(S)$. Since K is integral over k, and S is integrally closed, $K \subseteq S$, and hence $K \subseteq S/M$, where M is the maximal ideal of S. Since $S/M = k$, we get a contradiction.

By flatness, $S \otimes_k L \subset F(S) \otimes_k L$, so to prove the claim it suffices to see that $F(S) \otimes_k L$ is a domain. However, since k is algebraically closed in $F(S)$, and L is a separable extension of k, we may apply [Ja, Cor. 1, pg. 203] to obtain this conclusion. □

Corollary 4.5. *Let A and B be complete local rings containing a separably closed field k, which is the residue field of both A and B. Let* $P \in \text{Spec}(A)$, *and* $Q \in \text{Spec}(B)$. *Then* $\text{nilrad}(P \hat{\otimes}_k B + A \hat{\otimes}_k Q)$ *is a prime ideal in* $A \hat{\otimes}_k B$.

Proof. Both A and B are homomorphic images of power series rings over k; to prove the corollary we may assume A and B are equal to $k[[x_1, \ldots, x_d]]$, since the Corollary only depends on the quotients A/P and B/Q. Since k is a field, $(A/P) \hat{\otimes}_k (B/Q) \subset (A/P) \hat{\otimes}_k F(B/Q) = F(B/Q)[[x_1, \ldots, x_d]]/PF(B/Q)[[x_1, \ldots, x_d]]$, so to prove the statement, it is enough to show that $\text{nilrad}(F(B/Q)[[x_1, \ldots, x_d]]/PF(B/Q)[[x_1, \ldots, x_d]])$ is prime. Since k is separably closed, Corollary 4.3 proves this. □

5. Vanishing in regular local rings

The purpose of this section is twofold. First we simply collect our results of the previous section in the case that the base ring is regular; this is for the convenience of readers. We give a result and examples to show that some of the results are the best possible. The main point of this section, however, is to prove a theorem which gives a sufficient (but not necessary) criterion for $cd(A, I) \leqq d - 3$, where A is a regular local ring of dimension d. Recall that in Theorem 2.9 we gave a new proof of theorems of Hartshorne and Ogus (in char. 0) and Peskine and Szpiro (in char. p) which gave necessary and sufficient conditions for $cd(A, I) \leqq d - 2$. Our criterion in this section chiefly says that if every component of I has height at most $(d - 2)/2$, and no three minimal primes of I add to an m-primary ideal, then $cd(A, I) \leqq d - 3$. This criterion allows us to give a complete answer to the question if the number of components of I is at most 5 and we assume $\text{bight}(I) \leqq (d - 2)/2$. We begin, however, by collecting previous work.

Theorem 5.1. *Let A be an excellent regular local ring of dimension d containing a field, and I an ideal of A of bight$(I) = b$. Then:*

i) $cd(A, I) \leqq d - [(d - 1)/b]$.

ii) If, moreover, I is formally geometrically irreducible, then

$$cd(A, I) \leqq d - 1 - [(d - 2)/b] .$$

iii) If I is formally geometrically irreducible and $(A/I)_P$ is normal for all $P \supseteq I$ such that $\dim(A/P) \geqq 3$ *then.*

$$cd(A, I) \leqq d - [d/(b + 1)] - [(d - 1)/(b + 1)] .$$

Proof. Statement i) is due to Faltings [Fa1]. Statement ii) follows at once from (3.8), and iii) follows from (3.9). □

Theorem 5.2. *Let (A, m) be an excellent regular local ring of dimension d containing a field. Let $b < d$ be a positive integer. Set $s = [(d - 1)/b]$. Let $J_1, \ldots, J_s$ be ideals of A of big height b such that* $\text{Spec}(A/(J_1 + \ldots + J_s)) - \{m\}$ *is disconnected. Set $I = J_1 \cap \ldots \cap J_s$. Then $cd(A, I) = d - [(d - 1)/b]$.*

Proof. Set $\alpha_i = (\cap_{t \leq i} J_t) + J_{i+1} + \ldots + J_s$. We will prove by induction on i that $cd(A, \alpha_i) = d - i$. The theorem follows by setting $i = s$. If $i = 1$, the claim follows from (2.9). Assume that $i > 1$, and the claim has been shown for $i - 1$. Set $J' = J_i + \ldots + J_s$ and $J'' = (\bigcap_{t \leq i-1} J_t) + J_{i+1} + \ldots + J_s$. Then $\text{nilrad}(\alpha_i) = \text{nilrad}(J' \cap J'')$, and $\alpha_{i-1} = J' + J''$. By the lemma of [Ly1], $cd(A, J') < d - i + 1$, and $cd(A, J'') < d - i + 1$. Now the claim and the theorem follow from the Mayer–Vietoris sequence for the ideals J'' and J'', considering the induction hypothesis. □

Corollary 5.3 [Ly1]. *Let (A, m) be an excellent regular local ring containing a field. Let $I_0, \ldots, I_s$ be ideals of bigheight b such that $I_0 + \ldots + I_s$ is primary to the maximal ideal. Set $I = \bigcap_{j \leq s} I_j$. Then $cd(A, I) = d - s$, where s is as in Theorem 5.2.*

Proof. Set $J_1 = I_0 \cap I_1$, and $J_i = I_i$ for $i \geqq 2$. Apply Theorem 5.2. □

Theorem 5.2 covers examples not covered by (5.3). Such an example is:

Example 5.4. Let (A, m) be an excellent regular local ring containing a field.

Suppose $P_1, \ldots, P_5$ are primes of height b such that $P_i + P_k + P_j$ is primary to the maximal ideal iff $\{i, j, k\} \in \Lambda = \{(1, 3, 4), (1, 3, 5), (1, 2, 5), (2, 3, 4), (2, 4, 5)\}$. Set $I = P_1 \cap \ldots \cap P_5$. Note that Λ has the following property. Each i, $1 \leqq i \leqq 5$, appears no more than three times, and each pair $\{i, j\}$ appears no more than twice. This implies that we cannot split $\{1, \ldots, 5\}$ into three disjoint groups C, D, and E in such a way that the sum of $P_i + P_j + P_k$ is m-primary whenever $i \in C, j \in D$, and $k \in E$. Hence we cannot apply Corollary 5.3. However, we can split the ideal $I = J \cap K$ in such a way that $\operatorname{Spec}(A/(J + K)) - \{m\}$ is disconnected. Namely let $J = P_1 \cap P_2 \cap P_3$, and $K = P_4 \cap P_5$. We leave it to the reader to prove this claim.

Remark 5.5. The results of Theorem 5.1(i) and (ii) are exact for all d and b. Indeed, set $s = [(d-1)/b]$, and let $J_0, \ldots, J_s$ be ideals of A of bigheight $\leqq b$ such that $J_0 + \ldots + J_s$ is primary to the maximal ideal of A. Set $I = \cap J_i$. Then $cd(A, I) = d - s$. This follows from Corollary 5.3, and shows that (i) is the best possible.

To show that ii) is the best possible for all b and d with $0 < b < d$ requires more work. We separate this in:

Theorem–Example 5.6. Let k be an algebraically closed field. Set $R = k[x_1, \ldots, x_{d-b}]$, and set $s = [(d-1)/b]$. Let J be the defining ideal in R of $(s+1)$ closed points in general position in $\mathbb{A}_k^{d-b}$. By W denote the $(d-b)$-dimensional affine variety whose coordinate ring is $B = k \oplus J$. Embed W in $\mathbb{A}_k^n$ for some n, and let W^- denote the closure of W in $\mathbb{P}_k^n$. Let V be the image of W^- after some generic projection of W^- on $\mathbb{P}_k^d$ in $\mathbb{P}_k^n$. Let A be the homogeneous coordinate ring of $\mathbb{P}_k^d$ localized at the irrelevant ideal (of course $\dim(A) = d + 1$). Finally, let I be the defining ideal of V in A. Then I is formally geometrically irreducible of height b, and $cd(A, I) = d - s = \dim(A) - 1 - [(\dim(A) - 2)/b]$, i.e., the bound of (5.1) (ii) is exact for all such d and b.

Proof. Clearly B is a domain. We claim B is a finitely generated k-algebra. Let $M_1, \ldots, M_{s+1}$ be the maximal ideals of R such that $J = \cap M_i$. Since $R/M_i \cong k$ for each i, there exist $c_{ij} \in k$ with $x_j - c_{ij} \in M_i$. Then the elements $y_i = \Pi_{j=1,\ldots,s+1}(x_j - c_{ij}) \in J$, and R, and therefore B, is a finite module over $k[y_1, \ldots, y_{d-b}]$. Thus B is a finitely generated k-algebra of dimension $d - b$. By the same argument, R is integral over B, and is also clearly birational, with conductor J. It follows that B has an isolated singularity at the maximal ideal m of B generated by J, and that R is the integral closure of B.

There exist exactly $s + 1$ prime ideals in R lying over mR, namely $M_1, \ldots, M_{s+1}$. Hence the completion of B with respect to m (which we denote by $B^\wedge$) has $s + 1$ minimal prime ideals $J_1, \ldots, J_{s+1}$.

We claim that $B^\wedge/J_i$ is regular for each i, and that $J_i + J_j$ is $\hat{m}$ primary if $i \neq j$. Indeed $B^\wedge$ has an isolated singularity at $\hat{m}$, so $\operatorname{nilrad}(J_i + J_j)$ must be equal to $\hat{m}$. The extension of B to R is unramified, and therefore so is the extension of $(B^\wedge/J_i) \to (R_{M_i})^\wedge$. As the residue field is algebraically closed, the extension is etale and must be an isomorphism. Thus $B^\wedge/J_i$ is regular, as claimed.

Let $G = k[[Z_1, \ldots, Z_n]]$ be the completion of the coordinate ring of $\mathbb{A}_k^n$ at the maximal ideal defining the closed point $\{m\}$ of $\operatorname{Spec}(B)$. Let $L \subset G$ be the extension to G of the defining ideal of W in $\mathbb{A}_k^n$. From our calculations above, it follows that

$L = L_1 \cap \ldots \cap L_{s+1}$, where G/L_i is regular of dimension $d - b$, and $L_i + L_j$ is equal to the maximal ideal of G if $i \neq j$.

Let $Y_1, \ldots, Y_{d-b}$ be $d - b$ generic linear combinations of $Z_1, \ldots, Z_n$. Then $G/L_i \cong k[[Y'_1, \ldots, Y'_{d-b}]]$, where the $'$ means taking images in G/L_i. After a change of variables, we may assume that $Y_i = Z_i$, for $1 \leqq i \leqq d - b$.

Let $C = k[[Z_1, \ldots, Z_d]] \subset G$. We have that $k[[Z_1, \ldots, Z_{d-b}]] \subseteq C/(L_i \cap C) \subseteq G/L_i = k[[Z_1, \ldots, Z_{d-b}]]$, which forces $C/(L_i \cap C) \cong k[[Z_1, \ldots, Z_{d-b}]]$. Set $N_i = L_i \cap C$. From linear algebra it is clear that if $Z_{d-b+1}, \ldots, Z_d$ are chosen generically, then the fact that $L_i + L_j$ is to equal the maximal ideal of G forces $N_1 + \ldots + N_{s+1}$ to equal the maximal ideal of C.

Thus $I \subset A$ is formally geometrically irreducible (since I is a homogeneous prime), but upon localizing and completing at a prime ideal of height d, it splits as in Remark 5.5. This shows that $cd(A, I) = d - s$, as required. □

We now prove a lemma needed in the proof of the main theorem. (see [Fa2] and [Br–R]).

Lemma 5.7. *Let (A, m) be a complete regular local ring containing its residue field k, which is separably closed. If P and Q are two prime ideals of A, with $\dim(A/P) + \dim(A/Q) > d + 1 = \dim(A) + 1$, then $\operatorname{Spec}(A/(P + Q)) - \{m\}$ is connected.*

Proof. Write $A = k[[x_1, \ldots, x_d]]$. We may identify $\operatorname{Spec}(A/(P + Q))$ with $\operatorname{Spec}((A/P) \hat{\otimes}_k (A/Q)/\Delta)$, where Δ is the ideal generated by the diagonal in $A \hat{\otimes}_k A$. By Corollary 4.5, the ideal $\operatorname{nilrad}(P \hat{\otimes}_k A + A \hat{\otimes}_k Q) = N$ in $B = A \hat{\otimes}_k A$ is a prime ideal, and $\dim(B/N) = \dim(A/P) + \dim(A/Q) \geqq d + 2$. Since d is the number of generators of Δ, by [Fa2, Thm.6], we get the desired conclusion, since $\operatorname{Spec}((A/P) \hat{\otimes}_k (A/Q)/\Delta) - \{m\}$ is connected iff $\operatorname{Spec}(B/(N, \Delta)) - \{m\}$ is connected. □

Theorem 5.8. *Let (A, m) be a complete regular local ring containing its residue field, which is separably closed. Let I be an ideal of A, with $\{P_1, \ldots, P_n\}$ the minimal primes of I, $bight(I) = b$. Then $cd(A, I) \leqq d - 3$ if*

i) $2b + 2 \leqq d$ and,
ii) $P_i + P_j + P_k$ is not m-primary for any triple $\{i, j, k\}$.

Proof. We may assume that I is reduced, hence $I = \cap P_i$. We induct upon n to prove the theorem. First we consider the case $n = 1$. In this case, by Theorem 5.1 ii), $cd(A, I) \leqq d - 1 - [(d - 2)/b] \leqq d - 3$ as $2b + 2 \leqq d$.

Let $J = P_1 \cap \ldots \cap P_{n-1}$. By induction we may assume that $cd(A, J) \leqq d - 3$. Consider the Mayer–Vietoris sequence for local cohomology with respect to $J, P = P_n$, and $I = J \cap P$. We obtain that $H^q_I(M) \cong H^{q+1}_{J+P}(M)$ for all $q \geqq d - 2$ and for any A-module M, since $cd(A, J) \leqq d - 3$. Hence $cd(A, I) \leqq d - 3$ iff $cd(A, J + P) \leqq d - 2$. By Theorem 2.9, this is true iff $\dim(A/(J + P)) \geqq 2$, and $\operatorname{Spec}(A/(J + P)) - \{m\}$ is connected. First of all, $\dim(A/(J + P)) \geqq \dim(A) - ht(J) - ht(P) \geqq 2$ by our assumptions.

Suppose that $\operatorname{Spec}(A/(J + P)) - \{m\}$ is disconnected. Then there are ideals F and G such that $F + G$ is m-primary, $FG \subseteq \operatorname{nilrad}(J + P)$, but neither F nor G is contained in $\operatorname{nilrad}(J + P)$. Choose any P_i for $i \leqq n - 1$. By Lemma 5.7, $\operatorname{Spec}(A/(P_i + P)) - \{m\}$ is connected (since $\dim(A/P_i) + \dim(A/P) \geqq 2d - 2b \geqq d + 2$). Hence for each i, either F or G is contained in $\operatorname{nilrad}(P_i + P)$. If one of them, say F, is contained in $\operatorname{nilrad}(P_i + P)$ for every i, $1 \leqq i \leqq n - 1$, then $F \subseteq \cap(\operatorname{nilrad}$

$(P_i + P)) \subseteq \text{nilrad}(\cap (P_i + P)) \subseteq \text{nilrad}(J + P)$. This contradiction shows that $F \subseteq \text{nilrad}(P_i + P)$ for some i, and $G \subseteq \text{nilrad}(P_j + P)$ for some $j \neq i$. Then $m \subseteq \text{nilrad}(F + G) \subseteq \text{nilrad}(P_i + P_j + P_n)$, which contradicts ii). □

Remark 5.9. Theorem 5.8 recovers the special case of a general ideal given in Theorem 5.1 i). For $cd(A, I) \leqq d - [(d-1)/b]$, shows that $cd(A, I) \leqq d - 3$ provided $3b \leqq d - 1$, a condition that forces condition ii) of Theorem 5.8 to be true since $ht(P_i + P_j + P_k) \leqq 3b$ in general. Of course, (5.8) is more general than (5.1 i) since it only requires that $2b + 2 \leqq d$. One might expect similar better results for ideals of smaller codimension, but there are serious problems to extend the results further. Observe giving a necessary and sufficient criterion for $cd(A, I) \leqq d - 3$ is practically impossible. For let P be a prime with $\dim(A/P) = 3$. Then if P is a set-theoretic complete intersection, $cd(A, P) \leqq d - 3$, yet the vanishing of the obstructions to being a set-theoretic complete intersection are not sufficient to guarantee that $cd(A, I) \leqq d - 3$, as they are for the inequality $cd(A, I) \leqq d - 2$, where essentially being connected on the punctured spectrum guarantees the correct cohomological dimension.

Remark 5.10. It is certainly possible to have some triples of minimal primes add up to be m-primary, but none-the-less have the estimate $cd(A, I) \leqq d - 3$. For instance, the proof shows that at the inductive stage, not only must some triple add to be m-primary if the proof breaks down, but in fact one can divide the primes $P_1, \ldots, P_{n-1}$ into two disjoint sets, Λ = those containing F and Π = those containing G, such that given any $P_i \in \Lambda$, and any $P_j \in \Pi$, $P_i + P_j + P_n$ is m-primary. Although this phenomena is suseptible to analysis, we have not yet found a definitive statement.

To illustrate some of the problems with further analysis, a small amount of work yields a complete answer if the number of primes minimal over I is at most 5.

Corollary 5.11. *Let (A, m) be a complete regular local ring containing its residue field, which is separably closed. Let I be an ideal of A, with $\{P_1, \ldots, P_n\}$ the minimal primes of I, $bight(I) = b$. Assume that $2b + 2 \leqq d$.*

i) If $n \leqq 2$, then $cd(A, I) \leqq d - 3$.
ii) If $n = 3$, then $cd(A, I) \leqq d - 3$ iff $P_1 + P_2 + P_3$ is not m-primary.
iii) If $n = 4$ or 5, then $cd(A, I) \leqq d - 3$ iff whenever we divide $\{P_1, \ldots, P_n\}$ into two disjoint sets C and D, and let $J = \bigcap_{P \in C} P_j$ and $K = \bigcap_{P \in D} P$, $\text{Spec}(A/(J + K)) - \{m\}$ is connected. If $n = 4$, then $cd(A, I) \leqq d - 3$ iff there is at most one triple $\{i, j, k\}$ such that $P_i + P_j + P_k$ is m-primary.

Proof. If $n \leqq 2$, then the condition ii) of Theorem 5.8 is vacuously satisfied, so $cd(A, I) \leqq d - 3$. If $n = 3$, and $P_1 + P_2 + P_3$ is not m-primary then we can again apply Theorem 5.8 to obtain that $cd(A, I) \leqq d - 3$. This condition is also necessary by 5.3 with $s = 2$.

To prove (iii), first suppose that there exist two sets C and D as in the statement of (iii) such that $\text{Spec}(A/(J + K)) - \{m\}$ is disconnected. Then $H^{d-1}_{J+K}(A) \neq 0$. Since $H^{d-1}_J(A) = H^{d-1}_K(A) = 0$ (for dimensional reasons neither $\text{Spec}(A/J) - \{m\}$ nor $\text{Spec}(A/K) - \{m\}$ can be disconnected, so we may apply (2.9)) it follows from the Mayer–Vietoris sequence that $H^{d-2}_I(A) \neq 0$, so that $cd(A, I) = d - 2$. Hence the stated condition is necessary.

Assume the conditions in (iii). Certainly there must be some triple, say $\{1, 2, 3\}$,

such that $P_1 + P_2 + P_3$ is not m-primary (else the condition in (iii) is easily seen to fail). Set $J = P_1 \cap P_2 \cap P_3$, and $K = P_4$ if $n = 4$, and $K = P_4 \cap P_5$ if $n = 5$. By (i) and (ii) of this corollary, $cd(A, J) \leqq d - 3$, and $cd(A, K) \leqq d - 3$. By assumption $\operatorname{Spec}(A/(J + K)) - \{m\}$ is connected, so that $cd(A, J + K) \leqq d - 2$ by (2.9). Now the Mayer–Vietoris sequence shows that $cd(A, I) \leqq d - 3$. The second statement of iii) for $n = 4$ is pure combinatorics and is left to the reader. □

Example 5.12. Unfortunately, once the number of components of I is at least 6, the situation becomes much more complicated. A good case to consider is the following: let I be an ideal of $\operatorname{bight}(I) = b$, with $2b + 2 \leqq d$, and suppose that I has six minimal primes, $P_1, \ldots, P_6$. Set $\Lambda = \{\{1, 2, 3\}, \{1, 3, 4\}, \{2, 4, 5\}, \{2, 4, 6\}, \{1, 5, 6\}, \{3, 5, 6\}, \{2, 3, 5\}, \{3, 4, 6\}, \{1, 2, 6\}, \{1, 4, 5\}\}$. Suppose that $\{i, j, k\} \in \Lambda$ iff $P_i + P_j + P_k$ is m-primary. Is $cd(A, I) \leqq d - 3$? This example is arranged so that *no matter how one divides* I into two components J and K, either $cd(A, J)$ or $cd(A, K)$ is $> d - 3$. This prevents use of the Mayer–Vietoris sequence to compute $cd(A, I)$.

6. Topological applications

In this section we present some topological applications of our vanishing theorems, applied mainly to subschemes of projective space. Most of the translation from cohomological information to topological information was done by Grothendieck, Hartshorne [Ha1], [Ha2], and Ogus [O]. In particular we are able to show that for subvarieties of projective space, the homotopy groups are finite up to an integer depending upon the dimension of the ambient projective space, and the codimension of the variety. We begin by proving a proposition which is probably well-known to experts.

Proposition 6.1. *Let $X \subseteq \mathbb{P}^e(\mathbb{C})$ be a closed equidimensional algebraic set, and let $Y \subseteq X$ be a closed subset of X such that $X - Y$ is smooth. Let A be the homogeneous coordinate ring of X, and $I \subseteq A$ the defining ideal of Y. Then $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq \dim(X) - cd(A, I)$.*

Proof. The local cohomology exact sequence implies that the cohomological dimension of $\operatorname{Spec}(A) - V(I)$ is $cd(A, I) - 1$. Hence the cohomological dimension of $X - Y$ is also $cd(A, I) - 1$, since $\operatorname{Spec}(A) - V(I)$ is a $\mathbb{C}^*$ bundle over $X - Y$. Now [Ha2, p. 148, Thm 7.4] implies that $H^i(X; \mathbb{C}) \to H^i(Y; \mathbb{C})$ is an isomorphism if $i < \dim(X) - cd(A, I)$, and an injection if $i = \dim(X) - cd(A, I)$. By the universal coefficient theorem [Sp, pg. 244] this implies that the map $H_i(Y; \mathbb{C}) \to H_i(X; \mathbb{C})$ is an isomorphism for $i < \dim(X) - cd(A, I)$, and surjective if $i = \dim(X) - cd(A, I)$. This implies the result. □

Theorem 6.2. *Let $Y \subseteq \mathbb{P}^e(\mathbb{C})$ be a closed algebraic set consisting of irreducible components of codimension $\leqq b$. Then $H_i(\mathbb{P}^e(\mathbb{C}), Y; \mathbb{C}) = 0$ provided,*

i) (cf. [P]) $i \leqq [e/b] - 1$, *if no additional assumptions on Y are made,*
ii) $i \leqq [(e - 1)/b]$, *if Y is irreducible,*
iii) $i \leqq [(e + 1)/(b + 1)] + [e/(b + 1)] - 1$, *if Y is normal.*

Proof. Let A be the homogeneous coordinate ring of $\mathbb{P}^e(\mathbb{C})$, localized at the irrelevant ideal, and let $I \subseteq A$ be the defining ideal of Y. Then $\dim(A) = e + 1$. We

see that $c = c(I) = \text{embdim}(A) - \text{mdim}(A/I) \leqq e - b$. We may apply (3.8) to obtain that $cd(A, I) \leqq (e + 1) - [e/b]$, and hence by Proposition 6.1 the result (i) follows.

Consider (ii), using the same notation as in the proof of (i). Since I is formally geometrically irreducible, we can apply (3.8) to conclude that $cd(A, I) \leqq e + 1 - [(e - 1)/b]$. Applying Proposition 6.1 yields (ii).

Finally we prove (iii). Since I is formally geometrically irreducible, and A/I is normal on the punctured spectrum, we may apply (3.9) to obtain that $cd(A, I) \leqq (e + 1) - [(e + 1)/(b + 1)] - [e/(b + 1)]$. Applying Proposition 6.1 gives (iii). □

Remark 6.3. The result of (6.2 i) is exact for all e and b if b does not divide e. Let A be the homogeneous coordinate ring of $\mathbb{P}^e(\mathbb{C})$, and let $V \subseteq \mathbb{P}^e(\mathbb{C})$ be the union of $[e/b] + 1$ coordinate subspaces of codimension b in general position. Let I be the defining ideal of V in A. By (5.5) $cd(A, I) = (e + 1) - [e/b]$. By (5.1) i) for any prime P not equal to the irrelevant maximal ideal m, $cd(A_P, I_P) < e + 1 - [e/b]$. Thus $H^q_I(A)$ with $q = e + 1 - [e/b]$ is supported at m only. It follows from the proof of [O, Thm 4.4] that in this situation $H^q_I(A) \neq 0$ is equivalent to the fact that $H^{[e/b]}(\mathbb{P}^e\mathbb{C}) \to H^{[e/b]}(V; \mathbb{C})$ is not an isomorphism. Since this map is known to be injective, we conclude that $H^{[e/b]}(\mathbb{P}^e, V; \mathbb{C}) \neq 0$. By the universal coefficient theorem, $H_{[e/b]}(\mathbb{P}^e, V; \mathbb{C}) \neq 0$. In fact 6.2i) is exact even if b divides e [Ly2].

Theorem 6.4. *Let $Y \subseteq \mathbb{P}^e(\mathbb{C})$ be a closed algebraic set consisting of irreducible components of codimensions $\leqq b$. Assume that Y can be cut out set-theoretically by t equations along every subvariety of $\mathbb{P}^e(\mathbb{C})$ of dimension $\geqq m$. Then $H_i(X, Y; \mathbb{C}) = 0$ for*

i) $i \leqq e - t - b + 1$ if $m = 1$,
ii) $i \leqq e - t - b + 1$, if Y is irreducible and $m = 2$.

Proof. The proof follows at once from (6.1) and (3.2) in case i), and from (6.1) and (3.10) in case ii). □

Remark 6.5. Assume that Y is locally set-theoretically a complete intersection of codimension b except at finitely many closed points. From (6.4 i) we get that $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq e - 2b + 1$. Similarly, if Y is irreducible and the set of closed points at which Y is not a set-theoretic complete intersection has dimension $\leqq 1$, we get from (6.4 ii) that $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq e - 2b + 1$. This is a generalization of the famous result of Barth [Ba].

Before we are able to apply these results to show finiteness of relative homotopy groups, we need to know that varieties of low codimension are simply connected. The results of Fulton and Lazersfeld allow one to show:

Proposition 6.6. *Let $Y \subseteq \mathbb{P}^e(\mathbb{C})$ be a closed algebraic set consisting of irreducible components of codimension $\leqq e/3$. Then Y is simply connected.*

Proof. Let $Y_1, \ldots, Y_r$ be the irreducible components of Y. We use induction on r to prove the proposition. If $r = 1$ the result follows from [F-L, Cor. 5.3, p. 47]. By induction $Y' = Y_1 \cap \ldots \cap Y_{r-1}$ is simply connected. The intersection $Y' \cap Y_r$ is the union of $(r - 1)$ algebraic sets $Y_i \cap Y_r$, each one of which is connected by [F-H, Cor. 1], since $\text{codim}(Y_i) + \text{codim}(Y_r) \leqq e - 2$. The intersection of any two of them,

say $(Y_i \cap Y_r) \cap (Y_j \cap Y_r) = Y_i \cap Y_j \cap Y_r$ is nonempty as the sum of the codimensions of any three of the Y_k is at most e. Therefore $Y' \cap Y_r$ is connected, and the proposition follows from the Van Kampen theorem. □

Theorem 6.7. *Under the hypotheses of* (6.2) *and* (6.4), $\pi_i(\mathbb{P}^e(\mathbb{C}), Y)$ *is finite in the same range as* $H_i(X, Y; \mathbb{C}) = 0$, *except in* (6.4 *i*) *where we need the additional assumption that* $b \leqq e/3$.

Proof. In (6.2 ii) and (6.4 iii), Y is simply connected by [F-L, Cor. 5.3, p. 47], if $b < e/2$, while in (6.4 i) it is simply connected by (6.6) (also in (6.2 i) provided $b \leqq e/3$). Now (6.2) and (6.4) imply that $H_i(X, Y; \mathbb{Z})$ is finite in the same range where $H_i(X, Y; \mathbb{C})$ is zero, and since finite groups form a Serre class (see [Sp, Chap. 9, Sec. 6]), $\pi_i(\mathbb{P}^e(\mathbb{C}), Y)$ also are finite.

If $b \geqq e/2$ in (6.2 ii) and (6.4 iii), then $i \leqq 1$, and $\pi_1(\mathbb{P}^e(\mathbb{C}), Y) = 0$ for Y connected, as can be seen from the exact sequence $\{1\} \to \pi_1(\mathbb{P}^e(\mathbb{C}), Y) \to \pi_0(Y) \xrightarrow{\cong} \pi_0(\mathbb{P}^e(\mathbb{C}))$. If $e/2 \geqq b > e/3$ in (6.2 i), then $i \leqq 1$, and $\pi_1(\mathbb{P}^e(\mathbb{C}), Y) = 0$ for the same reason, since $b \leqq e/2$ implies that Y is connected. Finally if $b > e/2$, in (6.2 i) then $i \leqq 0$ and $\pi_0(\mathbb{P}^e(\mathbb{C}), Y) = 0$ for all Y. □

It is natural to conjecture that $\pi_i(\mathbb{P}^e(\mathbb{C}), Y) = 0$ in the same range. This will be proved in an upcoming paper by the second author of this paper [Ly2].

Remark 6.8. Using (3.11) one can also prove the following result:

Let $X \subseteq \mathbb{P}^e(\mathbb{C})$ be an equidimensional closed algebraic set of dimension d, and let $Y \subseteq X$ be a closed algebraic set consisting of irreducible components of dimension $\geqq d'$. Assume that $X - Y$ is smooth, and that the singular locus of X has dimension $\leqq d - d'$. Then

i) $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq [(d + d' - e)/(d - d')]$.
ii) In addition to the above hypotheses, assume that Y is irreducible, and the singular locus of X has dimension $\leqq d - d' + 1$. Then, $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq [(d + d' - e - 1)/(d - d')] + 1$.
iii) In addition to the assumptions of ii), assume that Y is normal. Then, $H_i(X, Y; \mathbb{C}) = 0$ for $i \leqq [(d + d' - e)/(d - d' + 1)] + [(d + d' - e - 1)/(d - d' + 1)] + 1$.

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S-unit equations over number fields

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Introduction

Let K be an algebraic number field of degree d. Denote by $M(K)$ the set of places of K and write $M_\infty(K)$ for the set of archimedean places of K. For $v \in M(K)$ denote by $|\ |_v$ the associated absolute value, normalized such that on $\mathbb{Q}$ we have $|\ |_v = |\ |$ (standard absolute value) if v is archimedean, whereas for v nonarchimedean $|p|_v = p^{-1}$ if v lies above the rational prime p. Writing d_v for the local degree $[K_v : \mathbb{Q}_v]$ we put $\|\ \|_v = |\ |_v^{d_v/d}$.

Let S be a finite subset of $M(K)$ containing $M_\infty(K)$ and having s elements. We call an element $x \in K$ an S-unit if

$$\|x\|_v = 1 \quad \text{for each } v \notin S. \tag{1.1}$$

Let $a_1, \ldots, a_n$ be nonzero elements in K. We will study the equation

$$a_1 x_1 + \ldots + a_n x_n = 1 \tag{1.2}$$

in S-units $x_1, \ldots, x_n$.

Mahler [6] in 1933 proved that for $n = 2$ and for $K = \mathbb{Q}$, *Eq.* (1.2) *has only finitely many solutions in S-units.* In his proof, he used his p-adic generalization of the Thue–Siegel theorem. *For general* $n \geqq 2$ independently of each other Dubois and Rhin [1] and Schlickewei [9] showed *in case* $K = \mathbb{Q}$ *that Eq.* (1.2) *has only finitely many solutions in S-units* $x_1, \ldots, x_n$ *that are relatively prime in pairs.* Later on independently of each other van der Poorten and Schlickewei [7] and Evertse [4] proved *for an arbitrary number field* K, *that* (1.2) *admits only finitely many solutions in S-units* $x_1, \ldots, x_n$, *such that*

$$\textit{no proper subsum } a_{i_1} x_{i_1} + \ldots + a_{i_k} x_{i_k} \textit{ vanishes}. \tag{1.3}$$

The example $2^k - 2^k + 1 = 1$ shows that in fact (1.3) is also a necessary condition.

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In both papers [7] as well as [4] the essential tool in the proof is the use of Schlickewei's p-adic generalization [8] of W. Schmidt's Subspace Theorem. As the underlying method in the Subspace Theorem is ineffective, it does not provide bounds for the *sizes* of the solutions x_i of (1.2). However the situation changes when one asks for bounds for the *number* of solutions of (1.2). Evertse [3], using the theory of hypergeometric functions, proved that *for* $n = 2$ *equation* (1.2) *has not more than*

$$3 \cdot 7^{d+2s}$$

solutions in S-units x_1, x_2. For $n \geqq 2$ Evertse and Győry [5] showed, that *the number of solutions of* (1.2), (1.3) *is bounded by a constant c that does not depend upon the coefficients* $a_1, \ldots, a_n$. Their method of proof however does not allow one to determine how the constant depends upon K, S and n.

Recently W. Schmidt [14] succeeded in giving a quantitative version of his Subspace Theorem. This was generalized by Schlickewei [10] to include p-adic valuations. Using this result, Schlickewei [11] proved that *for* $K = \mathbb{Q}$ *the number of solutions of* (1.2), (1.3) *is bounded by a constant c that depends only upon the dimension n and on the cardinality s of S.* In [12] Schlickewei extended the quantitative Subspace Theorem to number fields. The main result of that paper allows us now to give an explicit upper bound for the number of solutions of (1.2) in general.

Theorem 1.1. *Let K be a number field of degree d. Let* $a_1, \ldots, a_n$ *be nonzero elements of K. Suppose S is a finite subset of* $M(K)$ *of cardinality s, containing* $M_\infty(K)$. *Then the equation*

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = 1 \tag{1.4}$$

has not more than

$$(4sd!)\, 2^{36nd!s^6} \tag{1.5}$$

solutions in S-units $x_1, \ldots, x_n$ *such that no proper subsum* $a_{i_1} x_{i_1} + \ldots + a_{i_k} x_{i_k}$ *vanishes.*

It should be pointed out that the 36 in the exponent of (1.5) could be certainly improved. We tried to avoid painstaking estimates. However, notice that the bound (1.5) is rather uniform in K and in S. It only depends upon the degree of K and upon the cardinality of S. In the opposite direction, it was shown by Erdös, Stewart and Tijdeman [2] that there exist equations (1.4) that have more than $\exp(Cs^{\frac{1}{2}}/\log s)$ solutions.

In proving Theorem 1.1, it will be more convenient to consider the homogeneous S-unit equation corresponding to (1.4). In fact we shall prove that *given nonzero elements* $\alpha_1, \ldots, \alpha_{n+1}$ *in K, the equation*

$$\alpha_1 x_1 + \ldots + \alpha_n x_n + \alpha_{n+1} x_{n+1} = 0 \tag{1.6}$$

has not more than (1.5) *projective solutions* $\overline{(x_1, \ldots, x_{n+1})}$ *satisfying the following properties.* $\overline{(x_1, \ldots, x_{n+1})}$ *has a representative* $(x_1, \ldots, x_{n+1}) \in K^{n+1}$ *whose components* x_i *are S-units and no proper subsum* $\alpha_{i_1} x_{i_1} + \ldots + \alpha_{i_k} x_{i_k}$ *of* (1.6) *vanishes.* It is clear that this claim is equivalent with the assertion of Theorem 1.1.

Remark. It will be seen in the proof, that the $d!$ in our bound (1.5) has its origin in the fact, that we make a detour via the smallest normal extension L of $\mathbb{Q}$ containing

K. For the degree of L we have the estimate $[L:\mathbb{Q}] \leqq d!$. If however K itself is a normal extension of $\mathbb{Q}$, then it will follow from the proof below, that the terms $d!$ in (1.5) may be replaced by d. (For details cf. Theorem 2.1 and Corollary 2.2 below.) Thus, *if K is a normal extension of $\mathbb{Q}$ of degree d our bound* (1.5) *can be replaced by*

$$(4sd)^{2^{36nd}s^6}.$$

2. The subspace theorem

The main ingredient in the proof is the quantitative version of the Subspace Theorem for number fields, which was proved recently by Schlickewei [12]. For the convenience of the reader this result will now be quoted.

Given a vector $\boldsymbol{a} = (\alpha_1, \ldots \alpha_n) \in K^n$ we put for $v \in M(K)$

$$|\boldsymbol{a}|_v = \begin{cases} (|\alpha_1|^2 + \ldots + |\alpha_n|^2)^{\frac{1}{2}} & \text{if } v \text{ is archimedean} \\ \max\limits_{1 \leqq i \leqq n} |\alpha_i|_v & \text{if } v \text{ is nonarchimedean} . \end{cases}$$

and $\| \boldsymbol{a} \|_v = |\boldsymbol{a}|_v^{\frac{d_v}{d}}$.

We define the height of $\boldsymbol{a}$ by

$$H(\boldsymbol{a}) = \prod_{v \in M(K)} \| \boldsymbol{a} \|_v . \tag{2.1}$$

Given a linear form L with coefficients $\alpha_1, \ldots, \alpha_n$ in K we put $\| L \|_v = \| \boldsymbol{a} \|_v$ and $H(L) = H(\boldsymbol{a})$, where $\boldsymbol{a} = (\alpha_1, \ldots, \alpha_n)$.

Theorem 2.1. *Let L be a normal extension of $\mathbb{Q}$ of degree l. Let R be a finite subset of $M(L)$ of cardinality r. Suppose that for each $v \in R$ we are given n linearly independent linear forms $L_1^{(v)}, \ldots, L_n^{(v)}$ in n variables with coefficients in L. Let $0 < \delta < 1$. Consider the inequality*

$$\prod_{v \in R} \prod_{i=1}^{n} \frac{\| L_i^{(v)}(\boldsymbol{\beta}) \|_v}{\| L_i^{(v)} \|_v \| \boldsymbol{\beta} \|_v} < H(\boldsymbol{\beta})^{-n-\delta} . \tag{2.2}$$

There exist proper subspaces $S_1, \ldots, S_{t_1}$ of L^n with

$$t_1 = [(8r\ell)^{2^{34n^2r^6}\delta^{-2}}] , \tag{2.3}$$

such that every solution $\boldsymbol{\beta} \in L^n$ either lies in $\bigcup_{i=1}^{t_1} S_i$ or satisfies

$$H(\boldsymbol{\beta}) < \max\{(n!)^{\frac{9}{\delta}}, H(L_i^{(v)})^{\frac{9lnr}{\delta}} \ (i = 1, \ldots, n; v \in R)\} \tag{2.4}$$

Before we apply Theorem 2.1 to our problem, a few remarks seem to be appropriate. Initially we do not assume that our field K is a normal extension of $\mathbb{Q}$. However let L be the smallest normal extension of $\mathbb{Q}$ containing K. Then $[L:\mathbb{Q}] = l < d!$. We let R be the set of prime divisors $w \in M(L)$ such that there exists $v \in S$ with $w|v$. Given $v \in S$, let $w_1, \ldots, w_r$ be the extensions of v to L. Given $\alpha \in K$ it is well known that we have

$$\| \alpha \|_v = \| \alpha \|_{w_1} \ldots \| \alpha \|_{w_r} . \tag{2.5}$$

Therefore if we start out in Theorem 2.1 with K instead of L, with forms $L_i^{(v)}$ with coefficients in K and with a subset $S \subset M(K)$, we still may apply Theorem 2.1 with the set R as described above. Now R has cardinality $\leqq s(d-1)!$ if S has cardinality s. Taking this into account in (2.3), the bound changes into

$$[(8s(d-1)!\,d!)^{2^{34nd!\,(d-1!)^6 s^6 \delta^{-2}}}] < [(4sd!)^{2^{35nd!\,s^6\delta^{-2}}}] .$$

Moreover the bound in (2.4) changes into

$$H(\boldsymbol{\beta}) < \max\{(n!)^{\frac{9}{\delta}},\ H(L_i^{(v)})^{\frac{9ns(d!)^2}{\delta}} \ (i = 1, \ldots, n;\ v \in S)\}.$$

We have proved.

Corollary 2.2. *Let K be a number field of degree d. Let S be a finite subset of $M(K)$ of cardinality s. Suppose that for each $v \in M(K)$ we are given n linearly independent linear forms $L_1^{(v)}, \ldots, L_n^{(v)}$ in n variables with coefficients in K. Let $0 < \delta < 1$. Consider the inequality*

$$\prod_{v \in S} \prod_{i=1}^{n} \frac{\| L_i^{(v)}(\boldsymbol{\beta}) \|_v}{\| L_i^{(v)} \|_v \, \| \boldsymbol{\beta} \|_v} < H(\boldsymbol{\beta})^{-n-\delta} . \tag{2.6}$$

There exist proper subspaces $T_1, \ldots, T_{t_2}$ of K^n with

$$t_2 = [(4sd!)^{2^{35nd!\,s^6\delta^{-2}}}] \tag{2.7}$$

such that every solution $\boldsymbol{\beta} \in K^n$ either lies in the union $\bigcup_{i=1}^{t_2} T_i$ or satisfies

$$H(\boldsymbol{\beta}) < \max\{(n!)^{\frac{9}{\delta}},\ H(L_i^{(v)})^{\frac{9ns(d!)^2}{\delta}} \ (i = 1, \ldots, n;\ v \in S)\}. \tag{2.8}$$

It is this version of the Subspace Theorem we are going to use in the sequel.

3. Application of the subspace theorem

To apply the Subspace Theorem for the proof of Theorem 1.1, it will be convenient to make a change of variables. Put

$$y_i = \alpha_i x_i \quad (i = 1, \ldots, n+1) . \tag{3.1}$$

Then Eq. (1.6) may be rewritten as

$$y_1 + \ldots + y_{n+1} = 0 , \tag{3.2}$$

and we are asking for solutions y_i of (3.2) satisfying

$$\prod_{v \in S} \| y_i \|_v = \prod_{v \in S} \| \alpha_i \|_v \quad (i = 1, \ldots, n+1) . \tag{3.3}$$

Write $\mathbf{y} = (\mathbf{y}', y_{n+1}) = (y_1, \ldots, y_n, y_{n+1})$. In the sequel, when we speak about solutions $\mathbf{y}$, it is always meant that (3.2) and (3.3) are satisfied.

Equation (3.2) defines a subspace say U of K^{n+1} of dimension n.

Lemma 3.1. *There are*

$$t_3 = (n+1)^s . (4sd!)^{2^{35nd!\,4s^6}} \tag{3.4}$$

proper subspaces $U_1, \ldots, U_{t_3}$ *of* U, *such that for every solution* $\mathbf{y}$ *of* (3.2), (3.3) *the following holds true.*

Either $\mathbf{y}$ *lies in the union* $\bigcup_{i=1}^{t_3} U_i$ *or we have*

$$H(\mathbf{y}') \leqq \max\{(n!)^{18}, n^{9ns(d!)^2}, A^2 n^{3n+2}\} \tag{3.5}$$

where

$$A = \prod_{v \in S} (\| \alpha_1 \|_v \ldots \| \alpha_{n+1} \|_v) \cdot \prod_{v \notin S} (\max\{\| \alpha_1 \|_v, \ldots, \| \alpha_{n+1} \|_v\})^{n+1} \tag{3.6}$$

Proof. We want to apply Corollary 2.2. For $v \in S$ we consider the linear forms

$$\begin{cases} L_1^{(v)}(\mathbf{y}') = y_1 \\ \quad \vdots \\ L_n^{(v)}(\mathbf{y}') = y_n \\ L_{n+1}^{(v)}(\mathbf{y}') = y_1 + \ldots + y_n \,. \end{cases} \tag{3.7}$$

Notice, that in (3.7) any n different among the $n+1$ linear forms $L_i^{(v)}$ are linearly independent. Moreover we have for $i = 1, \ldots, n$ and for any $w \in M(K)$

$$\| L_i^{(v)} \|_w = 1 \quad \text{and} \quad H(L_i^{(v)}) = 1 \tag{3.8}$$

whereas

$$\| L_{n+1}^{(v)} \|_w = \begin{cases} 1 & \text{for nonarchimedean } w \in M(K) \\ n^{\frac{d_w}{2}} & \text{for archimedean } w \in M(K) \end{cases} \tag{3.9}$$

and therefore

$$H(L_{n+1}^{(v)}) = n^{\frac{1}{2}} \,. \tag{3.10}$$

In view of (3.2) and (3.3) we get for any solution $\mathbf{y}$

$$\begin{aligned} \prod_{v \in S} (\| y_1 \|_v \ldots \| y_{n+1} \|_v) &= \prod_{v \in S} (\| L_1^{(v)}(\mathbf{y}') \|_v \ldots \| L_{n+1}^{(v)}(\mathbf{y}') \|_v) \\ &= \prod_{v \in S} (\| \alpha_1 \|_v \ldots \| \alpha_{n+1} \|_v) \end{aligned} \tag{3.11}$$

To use Corollary 2.2, we now divide the solutions $\mathbf{y}$ into classes as follows. Given $v \in S$ and $\mathbf{y}$, define $i(v)$ by

$$|y_{i(v)}|_v = \max\{|y_1|_v, \ldots, |y_{n+1}|_v\} \,. \tag{3.12}$$

Given a tuple $(i(v))_{v \in S}$ with $1 \leqq i(v) \leqq n+1$, we denote by $K = K((i(v))_{v \in S})$ the class of solutions $\mathbf{y}$ giving rise to the tuple $(i(v))_{v \in S}$ as in (3.12). The number of classes K equals

$$(n+1)^s \,. \tag{3.13}$$

We now fix one class K with corresponding tuple $(i(v))_{v \in S}$. Put $I(v) = \{1, \ldots, n+1\} \backslash \{i(v)\}$. Then (3.11) implies

$$\prod_{v \in S} \prod_{i \in I(v)} \| L_i^{(v)}(\mathbf{y}') \|_v = \left(\prod_{v \in S} \| \alpha_1 \|_v \ldots \| \alpha_{n+1} \|_v \right) \left(\prod_{v \in S} \| y_{i(v)} \|_v \right)^{-1} \,. \tag{3.14}$$

We infer from (3.1) and (3.3) that

$$\| y_i \|_v = \| \alpha_i \|_v \quad \text{for any } i (1 \leqq i \leqq n+1) \text{ and for any } v \notin S .$$

Therefore we get

$$\prod_{v \notin S} \max_{1 \leqq i \leqq n+1} \| y_i \|_v = \prod_{v \notin S} \max_{1 \leqq i \leqq n+1} \| \alpha_i \|_v . \tag{3.15}$$

Combination of (3.14) and (3.15) yields

$$\prod_{v \in S} \prod_{i \in I(v)} \frac{\| L_i^{(v)}(\mathbf{y}') \|_v}{\max\limits_{1 \leqq j \leqq n+1} \| y_j \|_v} = A \Big(\prod_{v \in M(K)} \max_{1 \leqq i \leqq n+1} \| y_i \|_v \Big)^{-n-1} . \tag{3.16}$$

with A as given by (3.6).

Given $v \in S$ define $k(v)$ by

$$| y_{k(v)} |_v = \max_{i \in I(v)} \{ | y_i |_v \} . \tag{3.17}$$

Then by (3.2) and (3.12) we obtain

$$| y_{i(v)} |_v = | y_{k(v)} |_v \quad \text{for nonarchimedean } v \in S ,$$

and

$$| y_{k(v)} |_v \leqq | y_{i(v)} |_v \leqq n | y_{k(v)} |_v \quad \text{for archimedean } v \in S .$$

Therefore

$$\| y_{i(v)} \|_v = \| \mathbf{y}' \|_v \quad \text{for nonarchimedean } v \in S , \tag{3.18}$$

and

$$n^{-\frac{d_v}{d}} \| y_{i(v)} \|_v \leqq \| \mathbf{y}' \|_v \leqq n^{\frac{1}{2}\frac{d_v}{d}} \| y_{i(v)} \|_v \quad \text{for archimedean } v \in S . \tag{3.19}$$

Using (3.8), (3.9), (3.18), (3.19) we infer from (3.16) that

$$\prod_{v \in S} \prod_{i \in I(v)} \frac{\| L_i^{(v)}(\mathbf{y}') \|_v}{\| L_i^{(v)} \|_v \| \mathbf{y}' \|_v} \leqq A n^{\frac{3}{2}n+1} H(\mathbf{y}')^{-n-1} . \tag{3.20}$$

Assume that

$$H(\mathbf{y}') > A^2 n^{3n+2} , \tag{3.21}$$

then (3.20) implies (remember (3.10))

$$\prod_{v \in S} \prod_{i \in I(v)} \frac{\| L_i^{(v)}(\mathbf{y}') \|_v}{\| L_i^{(v)} \|_v \| \mathbf{y}' \|_v} < H(\mathbf{y}')^{-n-\frac{1}{2}} . \tag{3.22}$$

We are now in a position to apply Corollary 2.2 with $\delta = \frac{1}{2}$ and we may infer that there exist

$$t_4 = [(4sd!)^{2^{35nd!}\, 4s^6}] \tag{3.23}$$

subspaces $V_1, \ldots, V_{t_4}$ of K^n containing all solutions $\mathbf{y}'$ of (3.20) satisfying

$$H(\mathbf{y}') \geqq \max \{ (n!)^{18}, n^{9ns(d!)^2} \} . \tag{3.24}$$

Summing up over the different classes K introduces the factor $(n+1)^s$ from (3.13), and the Lemma follows from (3.23), (3.21) and (3.24).

4. A gap principle

Lemma 4.1. *Suppose* $0 < \gamma < 1$ *and* $q \in \mathbb{N}$ *are given. Then there exists a subset* M *of cardinality*

$$\leqq 2^{\frac{1}{1-\gamma}q}$$

of the set $\{(\Gamma_1, \ldots, \Gamma_q) | \Gamma_1 + \ldots + \Gamma_q = \gamma, \Gamma_i \geqq 0 (i = 1, \ldots, q)\}$ *with the following property. For every point* $\mathbf{x} = (x_1, \ldots, x_q)$ *in* $\mathbb{R}^q$ *having* $x_i \geqq 0$ *for each* i $(1 \leqq i \leqq q)$, *there exists a point* $\boldsymbol{\Gamma} = (\Gamma_1, \ldots, \Gamma_1) \in M$ *such that for each* $i (1 \leqq i \leqq q)$

$$x_i \geqq \Gamma_i (x_1 + \ldots + x_n)$$

holds true.

This is Lemma 5.1 of [12]. As mentioned there, it appears implicitly already in Mahler [6].

Lemma 4.2. *Let* B *be a quantity with*

$$B \geqq n^{9n} . \tag{4.1}$$

Put

$$C = A^{\frac{1}{2(n^2-1)}} \tag{4.2}$$

with A *as in* (3.6).

Then the solutions $\mathbf{y}$ *of* (3.2), (3.3) *with*

$$B \leqq H(\mathbf{y}') \leqq BC \tag{4.3}$$

are contained in the union of not more than

$$(n+1)^{2s} 2^{2n^2 s} \tag{4.4}$$

proper subspaces of U.

Proof. Given n solutions $\mathbf{y}_1, \ldots, \mathbf{y}_n$ of (3.2), (3.3), (4.3), we shall consider the matrix

$$\begin{pmatrix} y_{11}, \ldots, y_{1,n+1} \\ \vdots \qquad\quad \vdots \\ y_{n1}, \ldots, y_{n,n+1} \end{pmatrix} \tag{4.5}$$

where y_{ij} is the j-th component of $\mathbf{y}_i$. Denote by D_j the determinant of the matrix obtained from (4.5) by deleting the j-th column. Then (3.2) implies

$$D_j = D_1 \quad \text{for each } j (1 \leqq j \leqq n+1) . \tag{4.6}$$

It was shown in (3.11) that any solution $\mathbf{y}$ of (3.2), (3.3) satisfies

$$\prod_{v \in S} (\| y_1 \|_v \ldots \| y_{n+1} \|_v) = \prod_{v \in S} (\| \alpha_1 \|_v \ldots \| \alpha_{n+1} \|_v) . \tag{4.7}$$

Let $K = K((i(v))_{v \in S})$ the class of solutions $\mathbf{y}$ defined by (3.12). Remember the definition of $k(v)$ in (3.18). Each class K splits into

$$n^s \tag{4.8}$$

subclasses K', where a subclass K' consists of those $\mathbf{y}$ in K that give rise to a fixed tuple $(k(v))_{v \in S}$.

Fix a subclass K' and put for $v \in S$ $J(v) = I(v) \setminus \{k(v)\}$ with $I(v)$ as in Sect. 3. Then combining (4.7) with (3.20) we get in view of (3.3) for each $\mathbf{y}$ in K'

$$\prod_{v \in S} \prod_{i \in J(v)} \frac{\| y_i \|_v}{\max\limits_{1 \leqq j \leqq n+1} \| y_j \|_v} \leqq nA \left(\prod_{v \in M(K)} \max_{1 \leqq j \leqq n+1} \| y_j \|_v \right)^{-n-1} .$$

We observe that by (3.19), (3.20) this implies

$$\prod_{v \in S} \prod_{i \in J(v)} \frac{\| y_i \|_v}{\max\limits_{1 \leqq j \leqq n+1} \| y_j \|_v} \leqq n^{\frac{n+3}{2}} AH(\mathbf{y}')^{-n-1} . \tag{4.9}$$

In (4.9) each factor of the double product on the left hand side is bounded by 1. Therefore, there exist nonnegative real numbers $x_{iv} (v \in S,\ i \in J(v))$ with $\sum\limits_{v \in S} \sum\limits_{i \in J(v)} x_{iv} = 1$ and with

$$\frac{\| y_i \|_v}{\max\limits_{1 \leqq j \leqq n+1} \| y_j \|_v} = \left(\prod_{v \in S} \prod_{k \in J(v)} \frac{\| y_k \|_v}{\max\limits_{1 \leqq j \leqq n+1} \| y_j \|_v} \right)^{x_{iv}} (v \in S,\ i \in J(v)) .$$

We may apply Lemma 4.1 with $0 < \gamma < 1$ yet to be specified and with $q = s(n-1)$. Consequently and in view of (4.9) there exists a tuple (Γ_{iv}) with $v \in S$, $i \in J(v)$ in M having

$$\sum_{v \in S} \sum_{i \in J(v)} \Gamma_{iv} = \gamma, \qquad \Gamma_{iv} \geqq 0$$

and

$$\frac{\| y_i \|_v}{\max\limits_{1 \leqq j \leqq n+1} \| y_j \|_v} \leqq \left(\prod_{v \in S} \prod_{k \in J(v)} \frac{\| y_k \|_v}{\max\limits_{1 \leqq j \leqq n+1} \| y_j \|_v} \right)^{\Gamma_{iv}} \leqq \left(n^{\frac{n+3}{2}} AH(\mathbf{y}')^{-(n+1)} \right)^{\Gamma_{iv}} \tag{4.10}$$

for each pair (i, v) with $v \in S$, $i \in J(v)$.

Recall from Lemma 4.1 that the set M has cardinality

$$\leqq 2^{\frac{1}{1-\gamma} \cdot s(n-1)} . \tag{4.11}$$

We subdivide K' further into subclasses $K'' = K''((\Gamma_{iv}))$. K'' consists of those $\mathbf{y} \in K'$ that satisfy (4.10) for the fixed tuple (Γ_{iv}). We proceed to show that any n solutions $\mathbf{y}_1, \ldots, \mathbf{y}_n$ that lie in a fixed subclass K'' are linearly dependent.

For this purpose we will estimate the $(n \times n)$-subdeterminants of (4.5). We may assume that $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are ordered such that

$$H(\mathbf{y}_1') \leqq H(\mathbf{y}_2') \leqq \ldots \leqq H(\mathbf{y}_n') . \tag{4.12}$$

Then (4.10) implies

$$\frac{\| y_{ki} \|_v}{\max\limits_{1 \leqq j \leqq n+1} \| y_{kj} \|_v} \leqq \left(n^{\frac{n+3}{2}} AH(\mathbf{y}_1')^{-(n+1)} \right)^{\Gamma_{iv}} \tag{4.13}$$

for each $k (1 \leqq k \leqq n)$ and for each pair (i, v) with $v \in S$, $i \in J(v)$.

For $v \in S$ choose $l_v \in \{1, \ldots, n+1\}$, $l_v \notin J(v)$ and consider the determinant D_{l_v}. Using (4.13) we obtain

$$\| D_{l_v} \|_v \leqq \left(n^{\frac{n+3}{2}} AH(\mathbf{y}_1')^{-(n+1)} \right)^{\sum\limits_{i \in J(v)} \Gamma_{iv}} \prod_{k=1}^{n} \max_{1 \leqq j \leqq n+1} \| y_{kj} \|_v \tag{4.14}$$

for nonarchimedean $v \in S$ and

$$\| D_{i_v} \|_v \leqq (n!)^{\frac{d_v}{d}} \left(n^{\frac{n+3}{2}} A H(\mathbf{y}_1')^{-(n+1)} \right)^{\sum_{i \in J(v)} \Gamma_{i_v}} \prod_{k=1}^{n} \max_{1 \leqq j \leqq n+1} \| y_{kj} \|_v \tag{4.15}$$

for archimedean $v \in S$.

Combining (4.14), (4.15) and (4.6) we see that

$$\prod_{v \in S} \| D_1 \|_v \leqq n! \left(n^{\frac{n+3}{2}} A \right)^{\gamma} H(\mathbf{y}_1')^{-\gamma(n+1)} \prod_{v \in S} \prod_{k=1}^{n} \max_{1 \leqq j \leqq n+1} \| y_{kj} \|_v . \tag{4.16}$$

Now suppose $v \notin S$. Here we have by (3.1), (3.3)

$$\| y_i \|_v = \| \alpha_i \|_v . \tag{4.17}$$

Choose l_v such that $\| \alpha_{l_v} \|_v = \max_{1 \leqq i \leqq n+1} \| \alpha_i \|_v$. Then we get by (4.17)

$$\| D_{l_v} \|_v \leqq \prod_{\substack{i=1 \\ i \neq l_v}}^{n+1} \| \alpha_i \|_v \tag{4.18}$$

Using the relation $\prod_{k=1}^{n} \max_{1 \leqq j \leqq n+1} \| y_{kj} \|_v = \left(\max_{1 \leqq j \leqq n+1} \| \alpha_j \|_v \right)^n$, we infer from (4.18)

$$\| D_{l_v} \|_v \leqq \frac{\prod_{i=1}^{n+1} \| \alpha_i \|_v}{\left(\max_{1 \leqq j \leqq n+1} \| \alpha_j \|_v \right)^{n+1}} \prod_{k=1}^{n} \max_{1 \leqq j \leqq n+1} \| y_{kj} \|_v .$$

Taking the product over $v \notin S$, observing (4.6) and using the fact that $\prod_{v \in S} \prod_{i=1}^{n+1} \| \alpha_i \|_v = \left(\prod_{v \notin S} \prod_{i=1}^{n+1} \| \alpha_i \|_v \right)^{-1}$ we obtain

$$\prod_{v \notin S} \| D_1 \|_v \leqq \frac{\prod_{v \notin S} \prod_{k=1}^{n} \max_{1 \leqq j \leqq n+1} \| y_{kj} \|_v}{\left(\prod_{v \in S} \prod_{i=1}^{n+1} \| \alpha_i \|_v \right) \left(\prod_{v \notin S} \max_{1 \leqq j \leqq n+1} \| \alpha_j \|_v \right)^{n+1}}$$

$$= A^{-1} \prod_{v \notin S} \prod_{k=1}^{n} \max_{1 \leqq j \leqq n+1} \| y_{kj} \|_v . \tag{4.19}$$

The bounds in (4.16) and in (4.19) give together

$$\prod_{v \in M(K)} \| D_1 \|_v \leqq n! \, n^{\frac{n+3}{2}} A^{\gamma - 1} H(\mathbf{y}_1')^{-\gamma(n+1)} \prod_{v \in M(K)} \prod_{k=1}^{n} \max_{1 \leqq j \leqq n+1} \| y_{kj} \|_v .$$

With (3.19), (3.20) we conclude that

$$\prod_{v \in M(K)} \| D_1 \|_v \leqq n! \, n^{\frac{n+3}{2}} A^{\gamma - 1} H(\mathbf{y}_1')^{-\gamma(n+1)} n^n \prod_{k=1}^{n} H(\mathbf{y}_k')$$

$$< n^{4n} A^{\gamma - 1} H(\mathbf{y}_1')^{-\gamma(n+1)} \prod_{k=1}^{n} H(\mathbf{y}_k') . \tag{4.20}$$

We now choose

$$\gamma = \frac{2n+1}{2(n+1)} .$$

Then (4.20) implies in conjunction with (4.2), (4.3)

$$\prod_{v \in M(K)} \| D_1 \|_v < n^{4n} A^{-\frac{1}{2(n+1)}} H(\mathbf{y}_1')^{-n-\frac{1}{2}} . H(\mathbf{y}_1')^n . A^{\frac{1}{2(n+1)}}$$

$$= n^{4n} H(\mathbf{y}_1')^{-\frac{1}{2}} .$$

Since we may assume that $n \geqq 2$, (4.1), (4.3) yield

$$\prod_{v \in M(K)} \| D_1 \|_v < 1 .$$

Therefore $D_1 = 0$ and by (4.6) $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are linearly dependent.

We have shown that the solutions $\mathbf{y}$ in a subclass K'' defined via the tuple (Γ_{iv}) and the subscripts $i(v)$, $k(v)$ are contained in an $(n-1)$-dimensional subspace of the solution space U.

By (3.13), (4.8) and (4.11) the total number of subclasses is bounded by

$$(n+1)^s n^s 2^{2(n^2-1)s}$$

and the Lemma follows.

5. Small solutions

Lemma 5.1. *Let $D \geqq 1$. Then the number of one-dimensional subspaces in K^n having a basis vector $\mathbf{x}$ which satisfies*

$$H(\mathbf{x}) \leqq D \tag{5.1}$$

is bounded by

$$n\, 36^{n-1}\, 2^{2d(n-1)}\, D^{2d(n-1)} . \tag{5.2}$$

Proof. This is a consequence of Lemma 8B of W.M. Schmidt [14]. In fact, it suffices to count the number of points $\mathbf{x} = (x_1, \ldots, x_n)$ with at least on component equal to 1. Suppose $\mathbf{x} = (1, x_2, \ldots, x_n)$. Then

$$H(\mathbf{x}) \geqq \max_{2 \leqq i \leqq n} H((1, x_i)) . \tag{5.3}$$

Schmidt's Lemma says that the number of elements $x \in K$ having $H((1, x)) \leqq D$ is bounded by

$$36(2D)^{2d} .$$

Allowing $n-1$ in the exponent for the number of possible choices of the maximum in (5.3) and the factor n for the number of choices of the component that equals 1, the assertion follows.

Applying Lemma 5.1 with

$$D = n^{9ns(d!)^2}$$

we get at once.

Lemma 5.2. *The solutions* $\mathbf{y}$ *of* (3.2), (3.3) *with*

$$H(\mathbf{y}') \leqq n^{9ns(d!)^2} \tag{5.4}$$

are contained in the union of not more than

$$t_5 = n\,36^n\,2^{2dn}\,n^{18n^2sd(d!)^2} \tag{5.5}$$

proper subspaces of the solution space U.

The remaining solutions will be taken care of by means of the gap principle.

Lemma 5.3. *The solutions* $\mathbf{y}$ *of* (3.2), (3.3) *with*

$$n^{9n} \leqq H(\mathbf{y}') \leqq A^2\,n^{3n+2} \tag{5.6}$$

with A *as in* (3.6) *are contained in the union of not more than*

$$t_6 = 4(n^2-1)(n+1)^{2s}\,2^{2n^2s} \tag{5.7}$$

proper subspaces of the solution space U.

Proof. By Lemma 4.2, for each subinterval of the interval (5.6) of the shape $[E, A^{\overset{1}{2(n^2-1)}}E]$

$$(n+1)^{2s}\,2^{2n^2s}$$

subspaces will suffice. On the other hand the interval (5.6) may be covered by $4(n^2-1)$ such subintervals and the assertion follows.

6. The total number of subspaces

Lemma 6.1. *Let* U *be the* n*-dimensional subspace of* K^{n+1} *defined by Eq.* (1.6). *There are*

$$t_7 = 2(n+1)^s(4sd!)^{2^{35nd!}4s^6} \tag{6.1}$$

proper subspaces $V_1, \ldots, V_{t_7}$ *of* U *such that every solution* $\mathbf{x}$ *of* (1.6) *whose components are* S*-units is contained in the union* $\bigcup_{i=1}^{t_7} V_i$.

Proof. It suffices to add the numbers t_3, t_5, t_6 from Lemmata 3.1, 5.2, 5.3, since the bounds given in (3.5), (5.4), (5.6) complement each other. We obtain

$$t_5 \leqq n^{18n^2\,s(d!)^3}, \qquad t_6 \leqq 2^{4n^2s},$$

and

$$t_3 + t_5 + t_6 \leqq 2t_3 = 2(n+1)^s(4sd!)^{2^{35nd!}4s^6}.$$

7. Proof of the theorem

In this section, when we speak about solutions of (1.6), it is always tacitly assumed that the components are S-units.

We proceed by induction on n. Let $Z(n) = Z(n, d, s)$ be the constant in (1.5). We want to show that $Z(n)$ is an upper bound for the supremum over $\alpha_1, \ldots, \alpha_{n+1}$ of

the number of projective solutions of (1.6). It is clear that for $n = 1$ there is at most one projective solution and our bound in that case is more than generous.

We now assume $n > 1$ and the Theorem to be proved for all $n' < n$. Let V be one of the subspaces of Lemma 6.1. We shall first only study solutions $\mathbf{x}$ of (1.6) lying in V. Since V is a proper subspace of U, the solutions $\mathbf{x} \in V$ satisfy apart from (1.6) an equation

$$\sum_{i \in I} \beta_i x_i = 0 \tag{7.1}$$

with nonzero coefficients $\beta_i \in K$ and a subset I of $\{1, \ldots, n\}$ of cardinality $|I| \geqq 2$.

Let J be a nonempty subset of I and consider those solutions $\mathbf{x}$ for which

$$\sum_{i \in J} \beta_i x_i = 0 \tag{7.2}$$

but no proper subsum of (7.2) vanishes. Notice that we have

$$|J| \geqq 2 . \tag{7.3}$$

Since $|J| \leqq n$ we may apply the induction hypothesis to (7.2) and we see that (7.2) admits not more than $Z(|J| - 1)$ projective solutions. Now for each solution $\mathbf{x}$ of (1.6) with $\mathbf{x} \in V$ and with (7.2), the components x_i with $i \in J$ are of the shape $(x_i)_{i \in J} = (x\gamma_i)_{i \in J}$, where $(\gamma_i)_{i \in J}$ is one of the projective solutions of (7.2). Moreover x is an S-unit. Using this in (1.6) we obtain

$$\left(\sum_{i \in J} \alpha_i \gamma_i \right) x + \sum_{\substack{i=1 \\ i \notin J}}^{n+1} \alpha_i x_i = 0 . \tag{7.4}$$

By assumption, in (7.4) no subsum vanishes. Therefore the induction hypothesis says, that (7.4) admits not more than $Z(n + 1 - |J|)$ projective solutions with components say δ, δ_i $(i \notin J)$. Hence the points with components $\delta\gamma_i$ for $i \in J$ and δ_i for $i \notin J$ represent all projective solutions corresponding to our set J. Thus there are not more than

$$Z(|J| - 1) Z(n + 1 - |J|) \leqq Z(n - 1)^2$$

projective solutions in V arising from the set J.

The number of possibilities for our subset J is bounded by 2^n. Therefore V does not contain more than $2^n Z(n-1)^2$ projective solutions. Using the bound (6.1) for the number of subspaces we see that (1.6) has not more than

$$2^n Z(n-1)^2 t_7$$

projective solutions. We obtain

$$\begin{aligned} 2^n Z(n-1)^2 t_7 &\leqq 2^n (4sd!)^{2\,36(n-1)d^1 s^6\,2}\, 2(n+1)^s (4sd!)^{2^{35nd^1}\cdot 4s^6} \\ &< (4sd!)^{2\,36(n-1)d^1 2s^6} (4sd!)^{2^{35\cdot 5n\cdot d^1}\cdot 4s^6} \\ &< (4sd!)^{2^{36n\,d^1 s^6}} = Z(n) . \end{aligned}$$

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Smooth structures on algebraic surfaces with finite fundamental group

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In this paper we want to discuss the following

Conjecture. A compact non-singular algebraic surface with finite fundamental group has at least two smooth structures which are stable under blow ups.

This means that for each compact algebraic surface X there exists a homeomorphic smooth manifold Y such that $X \# s \cdot \overline{\mathbb{C}P^2}$ is not diffeomorphic to $Y \# s \cdot \overline{\mathbb{C}P^2}$ for all $s \geqq 0$. To consider smooth structures which are stable under blow ups is very natural from the point of view of algebraic geometry since one has only to study minimal surfaces. Note that the conjecture is false if one allows in addition (arbitrary) stabilization with $\mathbb{C}P^2$ ([W] in the 1-connected case and in general [G] or [Kr_1]).

The main result of this paper is the following.

Theorem. *Let G be a finite group. Then there is a constant $c(G)$ such that the conjecture holds for all algebraic surfaces X with $\pi_1(X) \cong G$, Euler characteristic $e(X) \geqq c(G)$ and $c_1^2(X) \geqq 0$.*

The condition $c_1^2 \geqq 0$ holds automatically for minimal surfaces with finite fundamental group ([BPV], p. 188). Note that for each finite group G there exist algebraic surfaces X with $\pi_1(X) \cong G$, $c_1^2(X) \geqq 0$ and arbitrarily large Euler characteristic. Let Y be a minimal model of an algebraic surface with fundamental group the symmetric group S_m [Sh, p. 402 ff]. Embed G into S_m for some m and let X be the corresponding covering over Y with $\pi_1(X) \cong G$. As noted above $c_1^2(Y) \geqq 0$ and thus $c_1^2(X) \geqq 0$ and we can make $e(X)$ arbitrarily large by choosing m appropriately.

In ([HK_1], Corollary 1.5), we proved that there are only finitely many homeomorphism types of closed oriented 4-manifolds with given finite fundamental group G and fixed Euler characteristic.

Thus we obtain:

Corollary. *Let G be a finite group. Then the conjecture holds for all but perhaps a finite number of homeomorphism types of minimal algebraic surfaces X with $\pi_1(X) \cong G$.*

Our conjecture and the partial results should be seen in the context of a conjecture of Friedman and Morgan which states that the natural map from algebraic surfaces modulo deformation equivalence to smooth 4-manifolds modulo orientation preserving diffeomorphism is finite-to-one [FM_2]. The proof of this conjecture was recently announced in [FM_3]. This result solves our problem in some important cases like for instance the $K3$-surfaces.

Our conjecture is also related to a conjecture of A. Van de Ven: algebraic surfaces of different Kodaira dimension are never diffeomorphic. If this is true, our conjecture holds for those algebraic surfaces which are homeomorphic to an algebraic surface of different Kodaira dimension. An interesting test case for Van de Ven's conjecture is the Barlow surface X which is homeomorphic to $Y = \mathbb{C}P^2 \# 8 \cdot \overline{\mathbb{C}P^2}$, but is of general type. Recently it was shown that $X \# s \cdot \overline{\mathbb{C}P^2}$ is not diffeomorphic to $Y \# s \cdot \overline{\mathbb{C}P^2}$ for all $s \geqq 0$ [Ko].

In Sect. 1 we summarize some known methods and results concerning the conjecture and in Sects. 2 and 3 we give the proof of our Theorem.

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1. Background

In this section we summarize some facts and results concerning the conjecture for specific fundamental groups.

The topological s-cobordism Theorem [Fr_2] reduces the homeomorphism classification for nice fundamental groups to finding topological s-cobordisms. On the other hand Donaldson introduced new diffeomorphism invariants ([Do_1], [Do_2]) which are closely related to algebraic geometry.

In particular Donaldson [Do_2] shows that a simply connected compact algebraic surface is essentially indecomposable.

Theorem 1.1 [Do_2]. *Let X be a 1-connected compact algebraic surface. Then X is not diffeomorphic to a connected sum $M_1 \# M_2$ unless M_1 or M_2 have negative definite intersection form.*

One can use this result in combination with Freedman's classification of 1-connected manifolds to prove the conjecture for 1-connected algebraic surfaces which are homeomorphic to decomposable smooth manifolds.

This shows that the conjecture holds for all 1-connected compact algebraic surfaces X which are not homeomorphic to K, $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# s \cdot \overline{\mathbb{C}P^2}$. A more detailed analysis of the Donaldson invariants leads to a proof of the conjecture for $\mathbb{C}P^2 \# 9 \cdot \overline{\mathbb{C}P^2}$ [FM_1], the Kummer surface [FM_3] as well as $\mathbb{C}P^2 \# 8 \cdot \overline{\mathbb{C}P^2}$ [Ko].

We summarize the present status of the conjecture for 1-connected surfaces as follows.

Theorem 1.2 ([Do_1], [Do_2], [Fr_1], [FM_1], [FM_3], [Ko]). *The conjecture holds for compact 1-connected algebraic surfaces which are not homeomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2 \# s \cdot \overline{\mathbb{C}P^2}$ and $s \leqq 7$.*

In the non simply-connected case more work must be done on the homeomorphism classification to apply these ideas.

Theorem 1.3 ([HK_2], [T]). *The conjecture holds for compact algebraic surfaces X in the following cases:*

i) $\pi_1(X) \cong \mathbb{Z}_k$, $k > 1$ *and odd*
ii) $\pi_1(X) \cong \mathbb{Z}_{2k}$ *and* $e(X) \neq 4$
iii) $\pi_1(X) \cong SL_2(F_p)$, $p = 3$ *or* $5 \pmod 8$, $c_1^2(X) \geqq 0$ *and* $e(X) > 12$

Remark. From the known results concerning our conjecture one gets the impression that the simpler the topology of a surface the harder the problem. The hardest case seems to be $\mathbb{C}P^2$. For $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ one has at least a candidate. Note that $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is the 2-fold branched covering of S^4 along the standard embedding of $\mathbb{R}P^2 \# \mathbb{R}P^2$ with normal Euler class 0 (with twisted coefficients). In [FKV] an infinite family of new knottings is constructed. By [Fr_1] each ramified covering is homeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. The same construction can be carried out starting from $\# 2r \cdot \mathbb{R}P^2 \subsetneq S^4$. If $r = 5$ the corresponding ramified covering is diffeomorphic to a Dolgachev surface which is not diffeomorphic to $\mathbb{C}P^2 \# 9 \cdot \overline{\mathbb{C}P^2}$. It would be somewhat surprising if the analogous family of knottings for $r = 1$ would lead to diffeomorphic ramified coverings.

In the situation of Theorems 1.2 and 1.3 the problem simplifies if one blows up an algebraic surface a few times. It is perhaps more realistic to study the following weakening of our conjecture.

Weak conjecture. For each algebraic surface with finite fundamental group there exists an r such that after r blow ups it has at least two smooth structures which are stable under further blow ups.

Theorems 1.2 and 1.3 imply the weak conjecture for all algebraic surfaces with finite cyclic fundamental group.

2. Proof of the theorem

Two 4-manifolds M_0 and M_1 are called stably homeomorphic if for some natural numbers r_0 and r_1, $M_0 \# r_0(S^2 \times S^2)$ is homeomorphic to $M_1 \# r_1(S^2 \times S^2)$. Note that $r_0 = r_1$ if and only if $e(M_0) = e(M_1)$. In [HK_2] we proved the following cancellation result

Theorem 2.1 ([HK_2; 1.3]). *Let M_0 and M_1 be stably homeomorphic closed smooth 4-manifolds with the same finite fundamental group and Euler characteristic. Suppose that M_0 is homeomorphic to $M_0' \# 2(S^2 \times S^2)$. Then M_0 is homeomorphic to M_1.*

We define a $\mathbb{Z}$-action on the set of stable homeomorphism classes of smooth closed oriented 4-manifolds with fixed fundamental group G by connected sum with r copies of the Kummer surface $K: (r, M) \mapsto M \# r \cdot K$. Here for $r < 0$, rK means $|r| \cdot (-K)$. Note that $K \# (-K)$ is homeomorphic to $\# 22(S^2 \times S^2)$. The next step in the proof is the following finiteness result which we will prove in the next section.

Proposition 2.2. *Let G be a finite group. The set of orbits of stable homeomorphism classes of smooth closed oriented 4-manifolds with fundamental group G under the action given by connected sum with K is finite.*

For each orbit choose a representative M_i with $e(M_i)$ minimal such that $-8 \leqq \operatorname{sign} M_i < 8$, and $M_i \cong M_i' \# 2(S^2 \times S^2)$. Then for each closed oriented smooth 4-manifold X with $\pi_1(X) \cong G$ there exist i and r such that X is stably homeomorphic to $M_i \# r \cdot K$. If $e(X) \geqq e(M_i \# r \cdot K)$ then X is homeomorphic to $M_i \# r \cdot K \# s(S^2 \times S^2) = Y$ for some s by Theorem 2.1. On the other hand Donaldson's Theorem (1.1) implies that if X is an algebraic surface, $\tilde{X}$ and $\tilde{Y}$ are not diffeomorphic (note that M_i decomposes as $M_i' \# 2(S^2 \times S^2)$). Thus the proof of our Theorem is finished if we can find a number $c(G)$ such that $e(X) \geqq e(M_i \# r \cdot K)$ for any algebraic surface X with $\pi_1(X) \cong G$, $e(X) \geqq c(G)$ and $c_1^2(X) \geqq 0$.

To compare $e(X)$ with $e(M_i \# r \cdot K)$ we express $e(M_i \# r \cdot K)$ in terms of $e(M_i)$, $\operatorname{sign}(M_i)$ and $\operatorname{sign}(X)$:

$$e(M_i \# r \cdot K) = e(M_i) + 22 \cdot |r|$$

and

$$\operatorname{sign}(X) = \operatorname{sign}(M_i \# r \cdot K) = \operatorname{sign} M_i - 16r.$$

This implies

$$e(M_i \# r \cdot K) = e(M_i) + (11/8)|\operatorname{sign}(X) - \operatorname{sign}(M_i)| \,.$$

Assume now that X is not simply-connected (see (1.2)). We have the following inequality:

$$e(X) - (11/8)|\operatorname{sign}(X)| \geqq (1/12)\, e(X) \,.$$

If $\operatorname{sign}(X) < 0$ this is an immediate consequence of the signature theorem and the assumption $c_1^2(X) \geqq 0$. If $\operatorname{sign}(X) \geqq 0$ then $c_1^2(X) > 0$ and X is of general type ([BPV], p. 188). We now apply the Miyaoka-Yau inequality [BPV, p. 207].

This implies together with the formula for $e(M_i \# r \cdot K)$ above (note that $\operatorname{sign}(X) \equiv \operatorname{sign}(M_i) \bmod 16$ and $-8 \leqq \operatorname{sign}(M_i) < 8$):

$$e(X) - e(M_i \# r \cdot K) = e(X) - (11/8)|\operatorname{sign}(X)| - e(M_i) \pm (11/8)\operatorname{sign}(M_i)$$

$$\geqq (1/12)\, e(X) - e(M_i) - 11 \,.$$

Thus, if $e(X) \geqq 12\, e(M_i) + 132$ we have $e(X) \geqq e(M_i \# r \cdot K)$.

As there are only finitely many M_i's we can define

$$c(G) := 12 \cdot \max\{e(M_i) + 11\}$$

finishing the proof of our Theorem.

3. Proof of proposition 2.2

The normal 1-type of a closed smooth oriented 4-manifold M is the fibre homotopy type of a fibration $\xi : B \to BO$ which is completely characterised by two facts: the homotopy groups of the fibre $\pi_r(F)$ vanish for $r \geqq 2$ and there is a lift of the normal Gauss map $\nu : M \to BO$ to B, denoted by $\bar{\nu} : M \to B$, which is a 2-equivalence (isomorphism on π_1 and surjective on π_2, [Kr_2]). In other words, the normal

1-type is the first stage of a Moore–Postnikov decomposition of the normal Gauss map. The pair $(M, \bar{\nu})$ represents an element in $\Omega_4(\xi)$, the bordism group of oriented 4-dimensional ξ-manifolds. The group of fibre homotopy self-equivalences $\mathrm{Aut}(\xi)$ operates on $\Omega_4(\xi)$ by composition and the class represented by $(M, \bar{\nu})$ in $\Omega_4(\xi)/\mathrm{Aut}(\xi)$ is independent of the choice of $\bar{\nu}$[Kr_2; p. 16]. The set of stable homeomorphism classes of smooth oriented closed 4-manifolds M having ξ as its normal 1-type is isomorphic to $\Omega_4(\xi)/\mathrm{Aut}(\xi)$ [Kr_2; 2.4].

The bordism group $\Omega_4(\xi)$ is finitely generated and $\Omega_4(\xi) \otimes \mathbb{Q} \cong H_4(B; \mathbb{Q})$ under $[M, \bar{\nu}] \mapsto \bar{\nu}_*[M]$ (compare [MS]). As $\pi_1(B)$ is finite the transfer $H_4(B; \mathbb{Q}) \to H_4(\tilde{B}; \mathbb{Q})$ is injective. Also $\tilde{B}$ is BSO if $w_2(\tilde{M}) \neq 0$ and BSpin if $w_2(\tilde{M}) = 0$. Thus $\Omega_4(\xi) \otimes \mathbb{Q} \cong \mathbb{Q}$ and the isomorphism is given by the signature.

Since the Kummer surface is 1-connected and almost parallelizable, connected sum with K does not change the normal 1-type of M. As $\mathrm{sign}(K) \neq 0$ we see that the orbit space of stable diffeomorphism classes of closed smooth oriented 4-manifolds M with normal 1-type ξ under the action of K is a finite set.

The proof of Proposition 2.2 follows from this if we show that for a fixed finite fundamental group G the number of normal 1-types of manifolds M with $\pi_1(M) \cong G$ is finite.

This is obvious if $w_2(\tilde{M}) \neq 0$ since then the normal 1-type is $k(\pi_1(M), 1) \times \mathrm{BSO} \xrightarrow{P_2} \mathrm{BSO}$. If $w_2(\tilde{M}) = 0$, let $f: Y \to M$ be a map inducing an isomorphism on π_1. Then it follows immediately from the characterization of a Moore–Postnikov factorization that the normal 1-type is determined by $(Y, \nu \circ f)$. We can obtain Y as the geometric realization of a fixed presentation of $G \cong \pi_1(M)$. But the set of homotopy classes of maps $Y \to BO$ is finite.

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Studying links via closed braids IV: composite links and split links

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Summary. The main result concerns changing an arbitrary closed braid representative of a split or composite link to one which is obviously recognizable as being split or composite. Exchange moves are introduced; they change the conjugacy class of a closed braid without changing its link type or its braid index. A closed braid representative of a composite (respectively split) link is composite (split) if there is a 2-sphere which realizes the connected sum decomposition (splitting) and meets the braid axis in 2 points. It is proved that exchange moves are the only obstruction to representing composite or split links by composite or split closed braids. A special version of these theorems holds for 3 and 4 braids, answering a question of H. Morton. As an immediate Corollary, it follows that braid index is additive (resp. additive minus 1) under disjoint union (resp. connected sum).

1. Introduction

This paper is part of a series of papers ([B-M, I, II, III, IV and V] in which the authors study representations of oriented links in oriented S^3 by closed braids. The long-term goal is to produce computable link type invariants by making use of one of the known solutions to the conjugacy problem in the braid group (e.g. [G]). Any such effort must deal with the various mechanisms which produce more than one conjugacy class of braids in a braid group B_n which represent the same link type. Two such mechanisms are treated in this paper.

A link K is *split* if there is a 2-sphere $\mathbf{X}$ in S^3-K which does not bound a 3-ball. A closed braid representative $\mathbf{K}$ of a split link K is a *split closed braid* if the splitting 2-sphere $\mathbf{X}$ can be chosen to meet the braid axis $\mathbf{A}$ in exactly two points. The closed 4-braid α_1 in Fig. 1 is an example of a split closed 4-braid which represents the disjoint union of two trefoil knots.

Since every link can be represented by a closed braid, it is obvious that every split link can be represented by a split closed braid. For, if K is the disjoint union of

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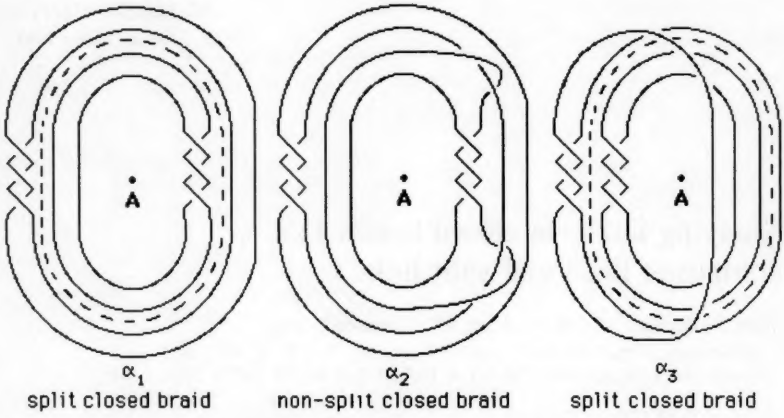

Fig. 1. Closed 4-braid representatives of the disjoint union of two trefoils

links $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$, choose an n_1-braid representative $\mathbf{K}_1$ of $\boldsymbol{K}_1$ and an n_2-braid representative $\mathbf{K}_2$ of $\boldsymbol{K}_2$ and use them to construct the obvious $(n_1 + n_2)$-braid representative $\boldsymbol{K}$ by banding together their respective axes. There is, however, a more subtle question, i.e. are there n-braid representatives of split links which are not isotopic in the complement of the axis to split closed n-braids?

The answer is "yes", and an example is given by the closed 4-braid α_2 in Fig. 1, which also represents the disjoint union $\boldsymbol{K}$ of two trefoil knots. If α_2 were isotopic in the complement of the axis to a split 4-braid representative of $\boldsymbol{K}$, it would necessarily be conjugate to α_1 because there is a unique closed 2-braid which represents the trefoil, and so a unique split closed 4-braid which represents the disjoint union of two trefoils. Let the symbol "i" (resp. "i^{-1}") denote an elementary braid in which the i^{th} strand crosses once over (under) the $(i+1)^{\text{st}}$ strand. Then α_1 and α_2 may be described by the braid words:

$$\alpha_1 = 1^3 3^3 ,$$

$$\alpha_2 = 1^3 2^{-2} 3^3 2^2 .$$

Now, there is a homomorphism $\phi\colon B_4 \to B_3$ defined by $\phi(1) = \phi(3) = 1$, $\phi(2) = 2$. If α_1 and α_2 were conjugate in B_4 then $\phi(\alpha_1)$ would be conjugate to $\phi(\alpha_2)$ in B_3. Using the solution to the conjugacy problem in B_3 which is given in [Mu] we obtain unique representatives 1^6 for the conjugacy class of $\phi(\alpha_1)$ and $\Delta^2 1^2 2^{-2} 1^2 2^{-2}$ for the class of $\phi(\alpha_2)$. Thus α_1 and α_2 are not conjugate in B_4. By a theorem of Morton [Mo, 3] this shows that the corresponding closed braids are not isotopic in the complement of **A**. Infinitely many such examples can be obtained by iterating the winding process which is shown in Fig. 1.

Remark. The example α_3 in Fig. 1 is also a split closed 4-braid representative of $\boldsymbol{K}$. We give it to show that the intersection of the splitting 2-sphere with the plane of projection need not be disjoint from the projected image of **K**.

Similar phenomena arise when we consider composite links. A link $\boldsymbol{K}$ is *composite* if there is a 2-sphere **Y** in S^3 which meets the link in 2 points and

decomposes it into sublinks $\boldsymbol{K}_1 \# \boldsymbol{K}_2$, neither of which is an unknotted arc. A closed braid representative **K** of $\boldsymbol{K}$ is *a composite closed braid* if the 2-sphere **Y** can be chosen to meet the braid axis **A** in 2 points. An example is given in Fig. 2. One may ask whether every closed n-braid representative of a composite link is conjugate to a composite closed n-braid? That question has received some attention in the literature. Morton showed that every closed 3-braid which represents a composite link is conjugate to an obviously composite 3-braid. In the same paper he also gave an example of a closed 5-braid which represents a composite link, but is not conjugate to a composite 5-braid. He was unable to decide whether 4-braids resemble 3 or 5-braids in this regard.

We will show in this paper how an arbitrary braid representative of a split (or composite) link can be changed to a split (or composite) braid representative. To state our contributions, we need the concept of an *exchange move*. In Fig. 3 the labels n_j on the strands are weights, indicating n_j parallel strands. We allow any type of braiding on the $n_1 + n_2$ (resp. $n_1 + n_3$) strands in the boxes which are labeled X(resp. Y). Assume that the braid axis **A** is the z axis, and that the arc which is labeled n_3 lies in the $x - y$ plane. Up to isotopy of S^3, an exchange is defined to be an isotopy of **K** which moves the arc which is labeled n_2 from a position which is a little bit above (or below) the $x - y$ plane to a position which is a little bit below (or above) the $x - y$ plane, keeping the rest of **K** invariant. An exchange move takes n-braids to n-braids, but need not preserve conjugacy class because the isotopy of **K** in S^3 is in general not realizable in the complement of the axis **A**. The first two results in this paper show that exchange moves are the only obstruction to representing split (or composite) links by split (or composite) closed braids:

The split braid theorem. *Let* $\boldsymbol{K}$ *be a split link, and let* **K** *be an arbitrary closed n-braid representative of* $\boldsymbol{K}$. *Then there exists a split n-braid* $\mathbf{K}^{\bullet}$ *which represents* **K** *and a*

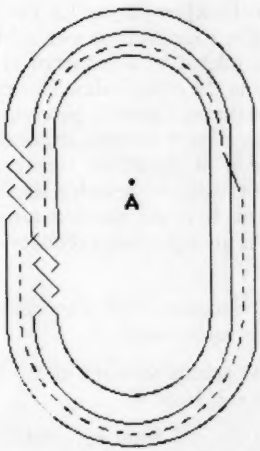

Fig. 2. Composite closed braid representing the connected sum of two trefoils

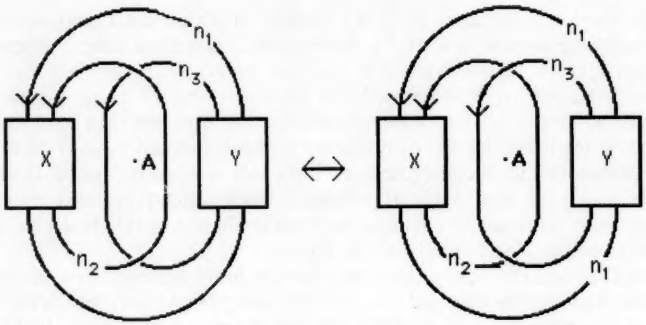

Fig. 3

finite sequence of closed n-braids:

$$\mathbf{K} = \mathbf{K}_0 \to \mathbf{K}_1 \to \mathbf{K}_2 \to \ldots \to \mathbf{K}_m = \mathbf{K}^*$$

such that each $\mathbf{K}_{i+1}$ *is obtained from* $\mathbf{K}_i$ *by either isotopy in the complement of the axis or an exchange.*

The composite braid theorem. *Let* K *be a composite link, and let* $\mathbf{K}$ *be an arbitrary closed n-braid representative of* K. *Then there exists a composite n-braid* $\mathbf{K}^*$ *which represents* K *and a finite sequence of closed n-braids*:

$$\mathbf{K} = \mathbf{K}_0 \to \mathbf{K}_1 \to \mathbf{K}_2 \to \ldots\ldots \to \mathbf{K}_m = \mathbf{K}^*$$

such that each $\mathbf{K}_{i+1}$ *is obtained from* $\mathbf{K}_i$ *by either isotopy in the complement of the axis or an exchange.*

The theorem of Markov ([Ma] or [Bi] or [Be] or [Mo, 2]) shows that, since the 4-braids α_1 and α_2 of Fig. 1 (or Morton's two 5-braids) represent the same oriented link type, they are related by a sequence of moves which involve first increasing and then decreasing the string index. This process is known as *stabilization*. Our theorems replace the stabilization process with a more direct process which changes one conjugacy class to another, preserving string index. It is thus a step in the proof of a version of "Markov's theorem without stabilization". A subsequent paper in this series will prove such a theorem in full generality.

The *braid index* of a link $\mathbf{K}$ is the smallest integer n such that $\mathbf{K}$ can be represented by an n-braid. As an immediate corollary of the Split and Composite Braid Theorems we obtain:

The braid index theorem. *Braid index is additive* (*respectively additive minus* 1) *under disjoint union* (*respectively connected sum*).

Our next result relates to special versions of the Split and Composite Braid Theorems which hold when $n = 3$ and 4.

The 3 and 4-braid theorem. (*i*) *Every 3-braid representative of a split link is conjugate to a split 3-braid representative. However, there are examples of 4-braid representatives of split links which are not conjugate to split 4-braids.*

(*ii*) *Every* 3*-braid representative of a composite link is conjugate to a composite* 3*-braid representative.*

(*iii*) *Let* $\mathbf{K}$ *be a* 4*-braid representative of a composite link* K*. If* K *cannot be represented by a closed braid of braid index* < 4*, then* $\mathbf{K}$ *is conjugate to a composite* 4*-braid. However, if* K *can be represented by a closed braid of braid index* < 4*, then* $\mathbf{K}$ *need not be conjugate to a composite* 4*-braid.*

2. Proof of the split braid theorem

Our work begins with a split link K, and an arbitrary oriented closed braid representative $\mathbf{K}$ of K. The braid axis is $\mathbf{A}$. We are also given a 2-sphere $\mathbf{X}$ which realizes the splitting of $\mathbf{K}$. We orient $\mathbf{K}$, and then orient $\mathbf{A}$ so that $\mathbf{K}$ is oriented in the positive sense about $\mathbf{A}$. We assign (arbitrarily) an orientation to $\mathbf{X}$, so that at each point of $\mathbf{X}$ there is a well-defined outward-drawn normal to $\mathbf{X}$. The 2-sphere $\mathbf{X}$ will in general be pierced by the braid axis $\mathbf{A}$ many times. Our goal is to modify $\mathbf{K}$ until there is a 2-sphere $\mathbf{X}'$ which realizes the splitting and is pierced twice by $\mathbf{A}$.

The principle tool in our work will be the study of the singular foliation of $\mathbf{X}$ which is induced by the fibration $\mathbf{H}$ of the open solid cylinder $R^3 - \mathbf{A}$ by half-planes $\{\mathbf{H}_t;\ t \in [0, 2\pi]\}$. The leaves of this foliation are the components of $\mathbf{X} \cap \mathbf{H}_t$, $t \in [0, 2\pi]$. The first step in the proof is to put $\mathbf{X}$ into a nice position relative to the fibration, in order to partially standardize the foliation.

By standard general position arguments we may assume:

(*i*) The intersections of $\mathbf{A}$ with $\mathbf{X}$ are finite in number and transverse.
(*ii*) There is a solid torus neighborhood $N(\mathbf{A})$ of $\mathbf{A}$ in S^3-$\mathbf{K}$ such that each component of $\mathbf{X} \cap N(\mathbf{A})$ is a disc.
(*iii*) The foliation in each component of $\mathbf{X} \cap N(\mathbf{A})$ is the standard radial foliation.
(*iv*) All but finitely many $\mathbf{H}_t$'s meet $\mathbf{X}$ transversally, and those which do not (the *singular fibers*) are each tangent to $\mathbf{X}$ at exactly one point in the interior of $\mathbf{H}_t$.
(*v*) The tangencies in (*iv*) are local maxima or minima or saddle points.

A *singular leaf* in the foliation of $\mathbf{X}$ will be one which contains a point of tangency. All other leaves are *non-singular*. Note that it follows from (*iv*) and (*v*) that:

(*a*) Each non-singular leaf is either an arc with both endpoints on $\mathbf{A} = \partial \mathbf{H}_t$ or a simple closed curve.
(*b*) A singular fiber $\mathbf{H}_\theta$ contains exactly one singular point p_θ.
(*c*) Each singular point p_θ is either a center or a saddle.

We now introduce a measure of the complexity of the pair $(\mathbf{X}, \mathbf{H})$. Let $|\mathbf{X} \cap \mathbf{A}|$ be the number of points in $\mathbf{X} \cap \mathbf{A}$. Let $|\mathbf{H} \cdot \mathbf{D}|$ be the number of singular points in the foliation of $\mathbf{X}$. The *complexity* $c(\mathbf{X}, \mathbf{H})$ is the pair $(|\mathbf{X} \cap \mathbf{A}|, |\mathbf{H} \cdot \mathbf{D}|)$. We assign the standard lexicographic ordering to this complexity function. We will say that $(\mathbf{X}, \mathbf{H})$ is *equivalent* to $(\mathbf{X}', \mathbf{H})$ if there is an isotopy taking $\mathbf{X}$ to $\mathbf{X}'$ which takes $(\mathbf{X} \cap \mathbf{H}_t, \mathbf{X} \cap \partial \mathbf{H}_t)$ to $(\mathbf{X}' \cap \mathbf{H}_t, \mathbf{X}' \cap \partial \mathbf{H}_t)$, for all $t \in [0, 2\pi]$. Notice that by our definition of equivalence, every representative of an equivalence class has the same complexity.

We investigate simple closed curves (SCC's) in $\mathbf{X} \cap \mathbf{H}$. We say that a pair $(\mathbf{X}, \mathbf{H})$ *has SCC's* if there exists a non-singular fiber $\mathbf{H}_t$ such that a component of $\mathbf{X} \cap \mathbf{H}_t$ is a SCC.

Lemma 1. *Assume that* $(\mathbf{X}, \mathbf{H})$ *satisfies* (i)–(v) *and has SCC's. Then there exists a splitting 2-sphere* $\mathbf{X}'$ *such that* $(\mathbf{X}^\bullet, \mathbf{H})$ *also satisfies* (i)–(v), *and in addition has no SCC's, moreover,* $c(\mathbf{X}^\bullet, \mathbf{H}) < c(\mathbf{X}, \mathbf{H})$.

Proof of Lemma 1. Our proof, sketched below, is very similar to the argument given in Lemma 2 of [B-M, 1]. The main difference is that our surface $\mathbf{X}$ is closed whereas the surface $\mathbf{D}$ in the proof of Lemma 2 of [B-M, 1] had non-empty boundary $\partial \mathbf{D} = \mathbf{K}$.

If there is a SCC $\alpha(t)$ in $\mathbf{X} \cap \mathbf{H}_t$ for some non-singular $\mathbf{H}_t$, we proceed as in the proof of Lemma 2 of [B-M, 1], following $\alpha(t)$ as it evolves in the fibration, until we arrive at a simple closed curve $\alpha(\theta)$ which contains a singularity of the foliation. Now $\alpha(\theta)$ lies on a singular fiber $\mathbf{H}_\theta$, and bounds a disc Δ in $\mathbf{H}_\theta$. Notice that Δ cannot be punctured by $\mathbf{K}$. For, the algebraic and geometric intersection numbers of $\mathbf{K}$ with Δ coincide, because Δ is a subdisc of $\mathbf{H}_\theta$ and $\mathbf{K}$ is a closed braid with axis $\partial \mathbf{H}_\theta$. There is also a second disc Δ', in $\mathbf{X}$, with $\partial \Delta = \partial \Delta'$. Since $\mathbf{K}$ does not intersect $\mathbf{X}$, it also does not intersect Δ'. Now, $\Delta \cup \Delta'$ forms a 2-sphere $\mathbf{S}$ in S^3, and the algebraic intersection number of $\mathbf{S}$ with $\mathbf{K}$ is zero because $\mathbf{S}$ is a 2-sphere. Since the geometric intersection number of Δ with $\mathbf{K}$ is equal to the algebraic intersection number of $\mathbf{K}$ with $\mathbf{S}$, it follows that $\mathbf{K}$ does not pierce Δ.

If the interior of Δ has empty intersection with $\mathbf{X}$, we surger $\mathbf{X}$ along Δ. The surgered surface $\tilde{\mathbf{X}}$ will be a pair of two 2-spheres, $\tilde{\mathbf{X}}(1)$ and $\tilde{\mathbf{X}}(2)$. At least one of these will be a new splitting 2-sphere, say $\tilde{\mathbf{X}}(1)$, and as in the proof of Lemma 2 of [B-M, 1] the new splitting 2-sphere will have complexity which is no more than that of the original $\mathbf{X}$. If both $\tilde{\mathbf{X}}(1)$ and $\tilde{\mathbf{X}}(2)$ are splitting 2-spheres, it won't matter which one we retain. If the interior of Δ meets $\mathbf{X}$, we find an innermost subdisc δ of Δ whose boundary is a component of $\mathbf{X} \cap \mathbf{H}_\theta$ and surger $\mathbf{X}$ along δ. Ultimately, we will arrive at a splitting 2-sphere $\mathbf{X}'$ which has the property that the induced foliation has no simple closed curves, and $c(\mathbf{X}^\bullet, \mathbf{H}) < c(\mathbf{X}, \mathbf{H})$. □

By Lemma 1, each component of intersection of a non-singular $\mathbf{H}_t$ with our 2-sphere $\mathbf{X}$ is an arc with both of its endpoints on $\mathbf{A} = \partial \mathbf{H}_t$. Such an arc divides $\mathbf{H}_t$ into two components, so it bounds two discs Δ and Δ' on $\mathbf{H}_t$. If either of these, say Δ, is *not* pierced by the link $\mathbf{K}$, we can push our 2-sphere $\mathbf{X}$ inward along a 3-space neighborhood of Δ to remove two points of intersection of $\mathbf{A}$ with $\mathbf{X}$. See Fig. 4. Of course, this could add simple closed curves to the foliation of $\mathbf{X}$, but the definition of our complexity function was chosen so that even if it does, the complexity will be reduced, because we will have reduced $|\mathbf{A} \cap \mathbf{X}|$. Moreover, even if SCC's are added, they can be removed with the aid of Lemma 1.

This motivates us to call a leaf β in $\mathbf{H}_t \cap \mathbf{X}$ *essential* if $\mathbf{K}$ pierces both components of $\mathbf{H}_t$ split along β. See Fig. 4 again. In view of the above remarks and Lemma 1, we can replace assumption (v) by the stronger assumptions:

$(v)^*$ The tangencies in (iv) are saddle points.
(vi) If $\mathbf{H}_t$ is non-singular, then each component of $\mathbf{X} \cap \mathbf{H}_t$ is essential.

We return to the main thread of the argument. The surface $\mathbf{X}$ is closed, so the braid axis $\mathbf{A}$ must pierce $\mathbf{X}$ an even number of times, say 2μ. If $2\mu = 2$, our closed braid is a split closed braid, and we are done, so assume $2\mu > 2$. Thus each component of intersection of $\mathbf{X}$ with a non-singular $\mathbf{H}_t$ is an arc which joins two of the 2μ points where the braid axis $\mathbf{A}$ pierces the 2-sphere $\mathbf{X}$. Recall that we assumed (see (iii) above) that the foliation of $\mathbf{X}$ is radial near each point of $\mathbf{A} \cap \mathbf{X}$. From this and the

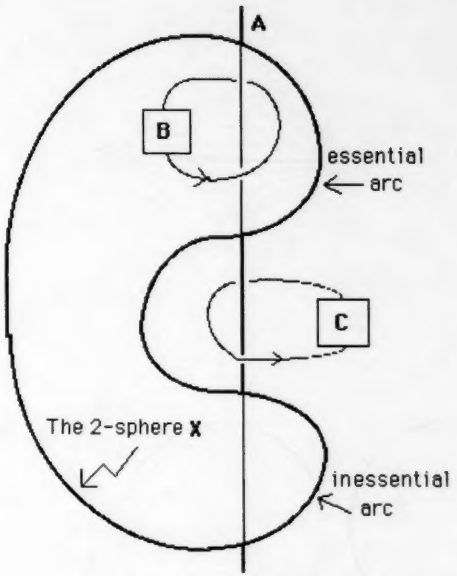

Fig. 4

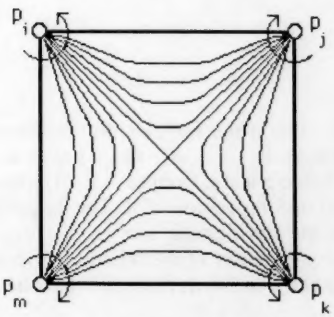

Fig. 5

fact that $2\mu > 2$ it follows that there must be singularities in the foliation. Each singularity is the result of a saddle-point type tangency of a singular fiber $\mathbf{H}_\theta$ with **X**. Figure 5 shows how the foliation looks in a neighborhood of a singular point. There will be four singular leaves which go out from the singular point like the spokes on a wheel, and end at points where the axis **A** pierces **X**.

Let $p_1, \ldots, p_{2\mu}$ be the points of $\mathbf{A} \cap \mathbf{X}$. Let $\theta_1, \ldots, \theta_q$ be the t-values at which singularities occur. The singular leaves are then the leaves in $\{\mathbf{H}_\theta \cap \mathbf{X};$

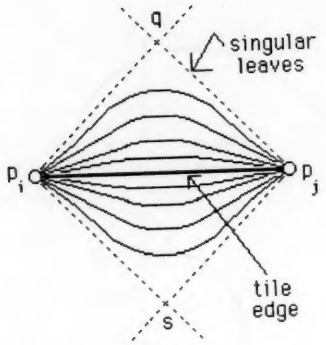

Fig. 6

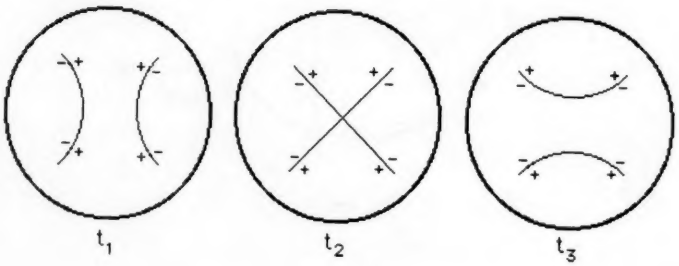

Fig. 7

$\theta = \theta_1, \ldots, \theta_q\}$ which contain the singular points. Their complement in $\mathbf{X}$ can only be a union of regions $R_1, \ldots, R_k$, as in Fig. 6, each of which is foliated without singularities. Choose a leaf e_j from each R_j. The union of all of these leaves, $\{e_1, \ldots, e_k\}$ gives a cell decomposition of $\mathbf{X}$. The 2μ points $p_1, \ldots, p_{2\mu}$ where $\mathbf{A}$ pierces $\mathbf{X}$ are the 0-cells, the non-singular arcs $\{e_1, \ldots, e_k\}$ are the 1-cells, and the 2-cells each contain one singularity of the foliation. Every 2-cell has four vertices and 4 edges (Fig. 5). We call our 2-cells *tiles*, and the foliated cell decomposition a *tiling* of $\mathbf{X}$.

The *sign* of a tile is defined as follows: each tile contains exactly one singularity of the foliation, which occurs at a point of tangency between $\mathbf{X}$ and a fiber $\mathbf{H}_\theta$ of $\mathbf{H}$. At the point of tangency the normals to $\mathbf{X}$ and $\mathbf{H}_\theta$ coincide. The sign of a tile is *positive* or *negative*, according as the orientations on the normals to $\mathbf{X}$ and $\mathbf{H}_\theta$ agree or disagree. Notice that if $(\theta - \varepsilon, \theta + \varepsilon)$ is a t-interval about θ which does not include any other singularities, then we can distinguish the two cases in the following way: Choose any $t \in (\theta - \varepsilon, \theta + \varepsilon)$ and label the sides of the arcs in $\mathbf{H}_t \cap \mathbf{X}$ "+" or "−" according as they correspond to the positive or negative sides of the oriented surface $\mathbf{X}$. Then as $t \to \theta$ at a positive (resp. negative) singularity the + (resp. −) sides of two arcs of $\mathbf{H}_t \cap \mathbf{X}$ will appear to coalesce, as in the left-to-right (resp. right-to-left) sequence in Fig. 7.

The *valence* of a vertex in the tiling is the number of tile edges which meet at that vertex. The next few lemmas concern the existence of vertices of low valence in the tiling. We will ultimately use these to recognize when our link admits a complexity-reducing exchange move.

Lemma 2. *The tiling of* **X** *always contains either a vertex of valence* 2 *or* 3.

Proof of Lemma 2. Let V, E and F denote the number of 0, 1 and 2-cells in the tiling. Then $V-E+F=2$, the Euler characteristic of **X**. Now, each 2-cell in the tiling has four edges in its boundary, and each edge is an edge of exactly two 2-cells, so that $2F = E$, and so $2V - E = 4$. Now let V_i be the number of vertices of valence i in the tiling. Since there are no vertices of valence 1, we have $V = V_2 + V_3 + V_4 + \ldots$. Since each edge has 2 vertices in its boundary, we also have that $2E = 2V_2 + 3V_3 + 4V_4 + \ldots$. Thus

$$2V_2 + V_3 = 8 + V_5 + 2V_6 + 3V_7 + \ldots.$$

The terms on both sides of this equation are non-negative. Thus there is always a vertex of valence 2 or 3. □

Lemma 3. *Either the tiling of* **X** *has a vertex of valence* 2, *or else there is an isotopy of* **X** *to a new splitting* 2*-sphere* **X**′, *such that the tiling of* **X**′ *has a vertex of valence* 2. *Moreover* $c(\mathbf{X}', \mathbf{H}) \leqq c(\mathbf{X}, \mathbf{H})$.

Proof of Lemma 3. If the tiling of **X** already has a vertex of valence 2 there is nothing to prove. If not, then by Lemma 2 the tiling has a vertex p of valence 3. There are 3 tiles which meet at this vertex (see Fig. 8a) and two of them (call them T

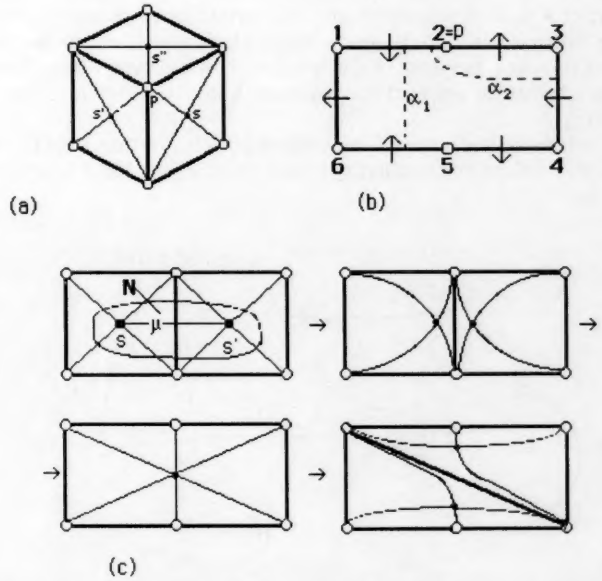

Fig. 8. **a** Vertex p has valence 3; **b** flow $T \cup T'$; **c** proposed changes in tiling $T \cup T'$

and T') necessarily have the same sign. Let their singular points be s and s'. Label the six vertices of $T \cup T'$ **1, 2, . . . , 6** in cyclic order, so that the vertex $p = \mathbf{2}$, as in Fig. 8b. Assume the origin to have been chosen so that the singularity at s occurs before the singularity at s'.

The future singularities have been indicated in Fig. 8b by two "joining arcs", labeled α_1 and α_2. We now explain what these mean. See Fig. 9, which shows the surface **X** and a fiber $\mathbf{H}_t$ of **H** just before one of the saddle point tangencies. We see two arcs, $\alpha \subset \mathbf{X}$ and $\beta \subset \mathbf{H}_t$. Together they bound a disc Δ in 3-space. These are our *joining arcs*. As $\mathbf{H}_t$ is pushed up the arcs α and β will shrink, vanishing at the instant of tangency. Now, in *our* situation there will be *two* singularities (at s and s'), and so there will be *two* pairs of joining arcs: α_1 and α_2 on **X** (illustrated in Fig. 8b) and β_1 and β_2 on $\mathbf{H}_t$ (illustrated in Fig. 10a). The idea of our proof is to show that we can deform **X** so that the two singularities are very close together both on **X** and in the fibration (we will make this precise below), and then interchange the order in which the singularities occur. The sequence of four pictures in Fig. 8c illustrate what we intend to do, from the point of view of an observer who is looking at the changing foliation of **X** during the deformation. After the deformation the tiling of $T \cup T'$ will have been changed so that the vertex p of valence 3 becomes a vertex of valence 2, also the tiling will be unchanged in $\mathbf{X} - T \cup T'$. We now show that these changes are always possible.

The first thing we want to prove is that the six tile vertices **1, 2, . . . , 6**, which are points on both **X** and **A**, must also have cyclic order **1, 2, . . . , 6** on **A**, as illustrated in the Fig. 10a, which depicts a fiber of **H** before the two singularities. There is no natural choice of a normal bundle on **X**, so we assign one arbitrarily so that **A** (oriented either way) pierces **X** from the + side at the vertex **1**. Note that this determines how **A** pierces **X** at every other vertex, because if the flow viewed from the + side of **X** is clockwise about any one vertex, then it is anticlockwise about any vertex which is joined to the given one by a leaf in the foliation. So, proceeding from vertex to vertex, the sense of the flow is determined everywhere. This, in turn, determines whether the oriented axis **A** pierces **X** from the + side or the − side at each vertex.

Let $\mathbf{ij}(t)$ denote a leaf in the foliation which joins the vertices **i** and **j**. In the given tiling of **X** (Fig. 8b) the first singularity is between $\mathbf{12}(t)$ and $\mathbf{56}(t)$. This shows that **1**

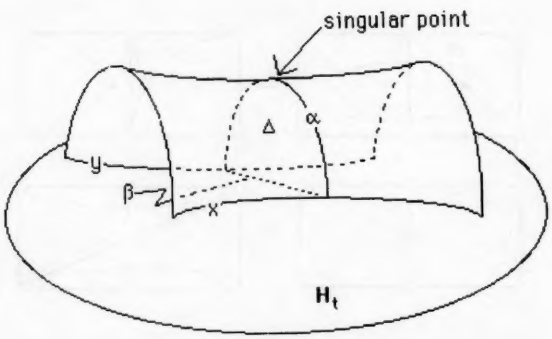

Fig. 9

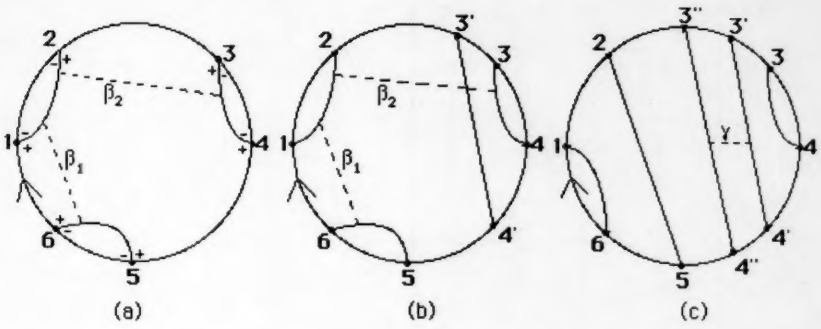

Fig. 10

and **2** cannot separate **5** and **6** on $\mathbf{A} = \partial \mathbf{H}_t$. Thus the cyclic order of the four vertices **1, 2, 5, 6** on the oriented curve **A** is **1256**, **2165**, **1265** or **2156**. However, the latter two are impossible because in the future surgery the sides of $\mathbf{12}(t)$ and $\mathbf{56}(t)$ which coalesce must have the same sign, and the sign is determined by our condition on the pierce-points. The first two choices are equivalent up to symmetry, and determined by whether the surgery is to be "+" or "−". We choose the order **1256**, dictating that the first surgery is to be a "+" surgery. Similar considerations apply to the pairs **2, 5** and **3, 4** giving two possible cyclic orders for the 6 points: **123456** and **125436**. However, with the latter choice the two singularities will have opposite signs, so the order must be **123456** as illustrated in Fig.10a.

We now have enough information to determine how $T \cup T'$ is embedded in 3-space. Choose 6 cyclically ordered points on **A** and declare them to be the tile vertices **1, 2, . . . , 6**. Then choose two fibers of **H** and declare them to be the fibers $\mathbf{H}_{\phi_1}$ and $\mathbf{H}_{\phi_2}$ which are to contain the singularities s and s' respectively. Then choose points s and s' in the fibers and declare them to be s and s'. Up to a homeomorphism of 3-space and a reparametrization of the fibers these choices are all arbitrary. Finally, join up s to the points **1, 2, 5, 6** by arcs in $\mathbf{H}_{\phi_1}$ which have that cyclic order when $\mathbf{H}_{\phi_1}$ is viewed from the + side, and similarly join s' to the points **2, 3, 4, 5** by arcs in $\mathbf{H}_{\phi_2}$, using that cyclic order when $\mathbf{H}_{\phi_2}$ is viewed from the + side. Thus we have embedded the singular leaves through s and s' in 3-space. The next thing to do is to adjoin little discs to the points **1, 2, . . . ,6**. These discs are to be the intersections of **X** with a neighborhood of **A** in 3-space. Since **A** pierces **X** from the + side (resp. − side) at **1, 3, 5** (resp. **2, 4, 6**), we know how to embed these discs as oriented surfaces. There is then a unique way to extend the embedding to a neighborhood of the singular leaves in 3-space. Finally, let μ be an arc which joins s to s' in **X**, and let **N** be a foliated neighborhood of μ, as in Fig. 8c. Then there is a unique way to adjoin **N** to the part of $T \cup T'$ already constructed. Finally, extend the embedding to all of $T \cup T'$. The result is depicted in Fig. 11a. (We have removed a tubular neighborhood of the axis **A** from 3-space, so the points **1, 2, . . . , 6** are replaced by circles, seen here as vertical arcs.) For comparison we have also shown the embedding when the signs disagree, in Fig. 11b.

As noted earlier, we want to push s and s' together on **X** and in the fibration and then to interchange their order in the fibration. There is clearly no obstruction to

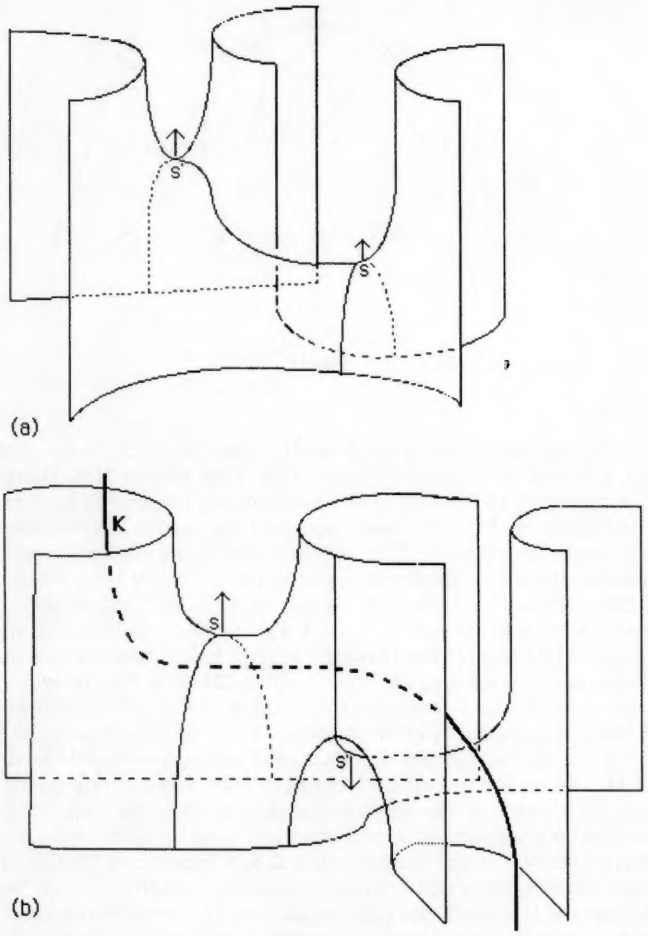

Fig. 11. **a** Non-degenerate saddles at s and s' signs at s and s' agree; **b** non-degenerate saddles at s and s' signs at s and s' do not agree

pushing s and s' close together on **X**. We now investigate whether there are obstructions in fibers of **H**. We first ask whether parallel sheets of **X** could interfere with the project (as they clearly might if the signs at s and s' disagree). Inspecting Fig. 10a, one potential difficulty becomes clear. It is possible that there is another leaf in the foliation of **X**, e.g. the leaf $\mathbf{3^*4^*}(t)$ illustrated in Fig. 10b, which is parallel to the leaf $\mathbf{34}(t)$ and obstructs the change we wish to make. It is not hard to see that, since we know that the singularity between $\mathbf{12}(t)$ and $\mathbf{56}(t)$ is possible, this is the only obstruction which can occur. After the first singularity the picture will be as in Fig. 10c. Now, since by hypothesis the singularity between $\mathbf{25}(t)$ and $\mathbf{34}(t)$ is

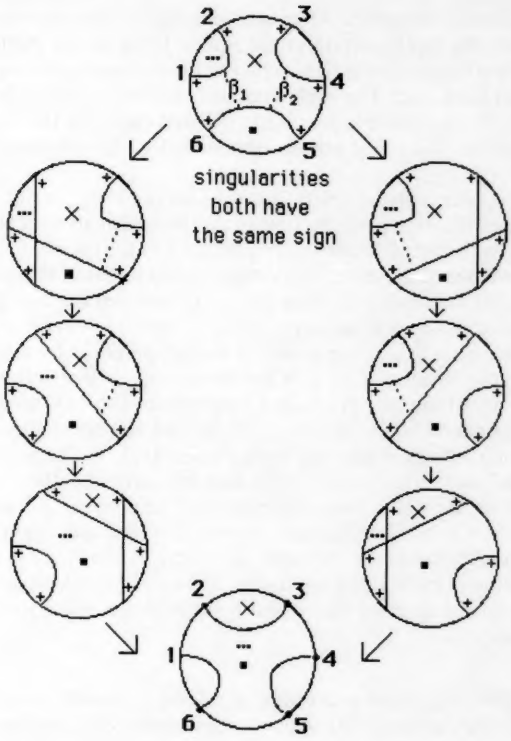

Fig. 12

possible, there must in fact be a *pair* of leaves which are parallel to **34**(t) in Fig. 10b, i.e. **3'4'**(t) and **3''4''**(t), and a singularity (indicated by the joining arc γ) between them which occurs *before* the singularity between **25**(t) and **34**(t). Now notice that on the surface **X** the four tile vertices **3'**, **3''**, **4'**, **4''** are disjoint from T and T' (and so not visible in Fig. 11a). Thus there is no obstruction to changing the fibration so that the singularity between **3'4'**(t) and **3''4''**(t) occurs *before* that between **12**(t) and **56**(t), removing the obstruction.

Since **K** intersects every fiber of **H**, we must also show that **K** cannot obstruct our proposed change in the order of the singularities. (Notice that **K** could be an obstruction in the situation of Fig. 11b, because if we were to push s down we would force **K** to be tangent to a fiber of **H**, so that it would no longer be a closed braid). Refer now to the sequence of pictures in Fig. 12. The top one is a repeat of Fig. 10a, with a little new data added. Initially, **K** will pierce one of the four regions of $\mathbf{H}_t$ split along the three arcs of $\mathbf{H}_t \cap \mathbf{X}$. By symmetry it suffices to consider three such choices. We have indicated these by a square black dot, a horizontal slash and an X in the three regions. We follow the time-evolution of the dot. It can wander anywhere within a region as t varies, but it cannot cross from one region to another

because **K** does not intersect **X**. After each singularity, new regions open up to it. We have shown the final position of the square black dot, if the left sequence is used. We now see that we could have achieved the same end position if, instead, the right route had been used. The same argument applies to the slash and to the X. For every possible position it is possible to proceed either via the left or instead via the right sequence. Thus **K** is not an obstruction to the change in order of the singularities.

The proof is essentially complete, however we can improve our understanding of the changes by showing precisely how to do the deformation of $\mathbf{N} \subset T \cup T'$. We may regard **N** as the graph of the smooth function $f(x, y) = y^3 - 3y(x^2 - 1)$. This function has two non-degenerate saddle-type critical points of the same sign, one at $(x, y) = (1, 0)$ and the other at $(x, y) = (-1, 0)$. Now, our function $f(x, y)$ belongs to a parametrized family of functions f_ε, with $f_\varepsilon(x, y) = y^3 - 3y(x^2 - \varepsilon^2)$ and $f = f_1$. If $\varepsilon \neq 0$, the function $f_\varepsilon(x, y)$ has a pair of critical points at $(\pm \varepsilon, 0)$. Both critical points are saddles because at $(\pm \varepsilon, 0)$ the Hessian is $-36\varepsilon^2 < 0$. Deform **X** to a new surface **X**′ by letting the parameter ε pass to zero. This has the effect of pushing s and s' toward one-another, until at $\varepsilon = 0$ the two saddles coalesce to a monkey saddle, which has a single degenerate critical point at (0, 0). The graph **X**′ of $f_0(x, y)$ has three "hills" and three "valleys". The singular leaves (i.e. the level sets $f_0^{-1}(0)$) are three lines in the $x - y$ plane through (0, 0), of slope 0, $\sqrt{3}$ and $-\sqrt{3}$. The surface **X**′ has local 3-fold rotational symmetry about the origin. We can then further deform **X**• by rotating it through, say $-\pi/\sqrt{3}$, allowing ε to increase again, and then rotating it back again, i.e. replace f_ε by $g^{-1} f_\varepsilon g$, where g is the rotation. This has the effect of splitting the critical point into two new critical points along the line of slope $\sqrt{3}$. □

Lemma 4. *If the tiling contains a vertex of valence* 2, *then* **K** *admits an exchange. Moreover, after the exchange the complexity can be reduced by removing two or more of the points where the axis pierces* **X**.

Proof of Lemma 4. There are two tiles which meet at our vertex of valence 2, and the first step in the proof is to examine the signs of the singularities in these two tiles. Let p be the vertex in the statement of the lemma, and let q and s be the singular points of the two tiles which meet at p. Let p' and p'' be the other vertices of the two tile edges which meet at p. The singular leaves through q and s occur at t_1 and t_5, respectively. Portions of these singular leaves fit together to cut off a disc δ on **X**, and the region depicted in Fig. 13 is a neighborhood Δ of δ on **X**, chosen so that $\partial \Delta$ is everywhere transverse to the foliation. We have labeled leaves of the foliation as occurring at times $t = t_1, t_2, \ldots, t_8$.

We now examine corresponding leaves as they would occur on a sequence of fibers of **H**. See the sequence of five pictures in Fig. 14, which are labeled t_3, t_4, t_5, t_6, t_7 to correspond to the labels in Fig. 13. The full cycle is obtained from this one by adding to it the corresponding sequence of pictures run backwards. The singularities which occur at t_1 and t_5 have opposite signs. (This is a direct consequence of the fact that there are exactly two singular leaves among the leaves which are incident at p). Thus the normal bundles to **X** and **H** agree (say) at s, and disagree at q. This is the first fact which we need, in order to see how Δ is embedded in 3-space.

The next thing to notice is that (from Fig. 5) the gradient flow on **X** is always oriented in opposite senses around the two endpoints of each leaf which joins two

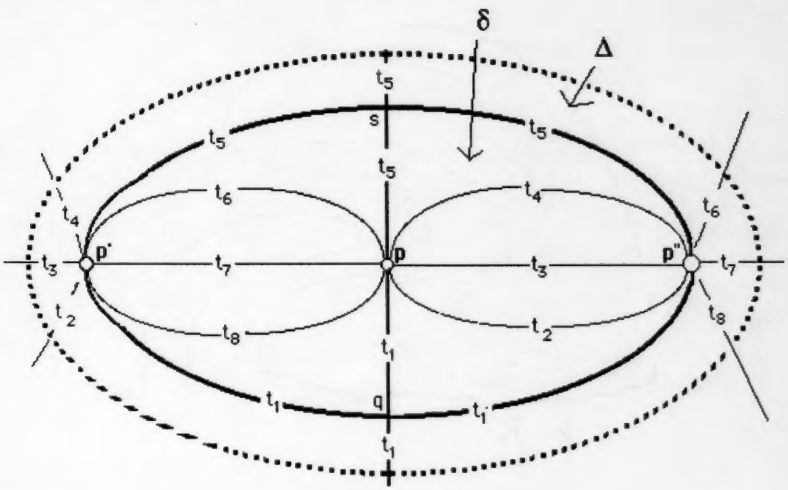

Fig. 13

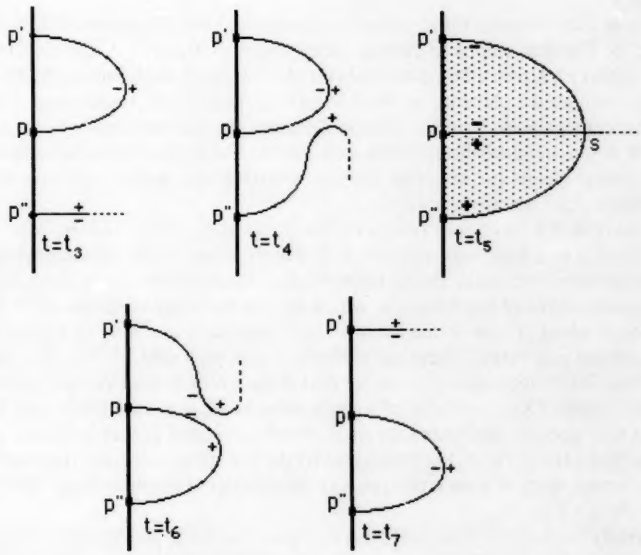

Fig. 14. Pictures in a cycle of fibers of H

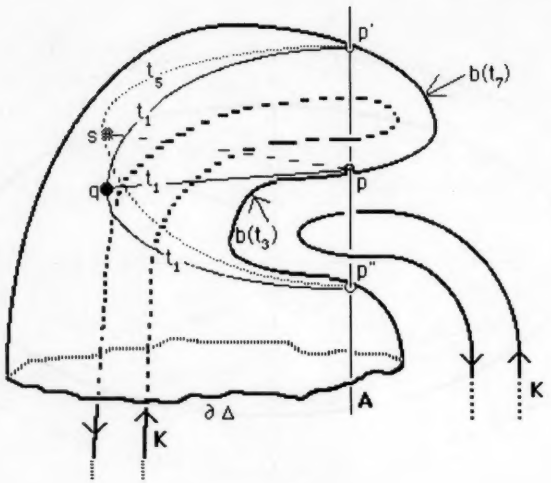

Fig. 15

points of $\mathbf{A} \cap \mathbf{X}$. This implies that $\partial \Delta$ has algebraic rotation number 1 about the axis **A**. Since $\partial \Delta$ is everywhere transverse to the foliation, it then follows that $\partial \Delta$ is a 1-braid, and also that the axis **A** pierces Δ from alternating sides at the 3 pierce-points, say the positive side, the negative side, and then the positive side again at p, p' and p''.

We now have enough information to determine how Δ is embedded in 3-space. See Fig. 15. The first step is to choose three points on the axis **A** and declare them to be p', p and p''. Up to a homeomorphism of S^3 which fixes **A** and each fiber it will not matter where we place them. Next, we attach small discs transverse to the axis with their centers at p', p and p''. These discs are the intersections of our oriented surface **X** with a neighborhood of the axis, and by the argument we have just given, if the disc at p' has its positive side up, then the discs at p and p'' will have negative and positive sides up, respectively.

The next step is to choose points on the fibers at t_1 and t_5 and declare them to be q and s. Up to a homeomorphism of S^3 which preserves **A** and each fiber it will not matter where we place them. Now, each singular point is the intersection of four singular leaves of the foliation, which go out from the singular point like the spokes on a wheel. Three of the leaves which meet at q and also at s terminate at p', p, p'', so we can extend them (in their fibers) to p', p and p''. The fact that the singularities have opposite signs shows that if the cyclic order (looking down onto the positive side of **X**) at q is p', p, p'', then it must be p'', p, p' at s. The fourth leaf at s and also at s' goes to the boundary of Δ, which (as noted earlier is a little 1-braid about **A**. Since the surface Δ is transverse to the foliation in the complement of the singular leaves, there is now a unique way to complete the embedding of Δ. In this way we obtain Fig. 15.

We study the non-singular leaf $b(t_7)$ in Figs. 12 and 13. By hypothesis, every leaf in the foliation is essential. This means that **K** itself is necessarily an obstruction to their removal, as illustrated in Fig. 15. We propose to remove this obstruction by

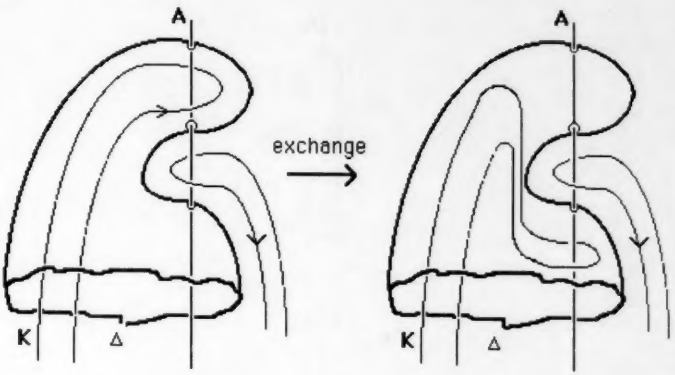

Fig. 16

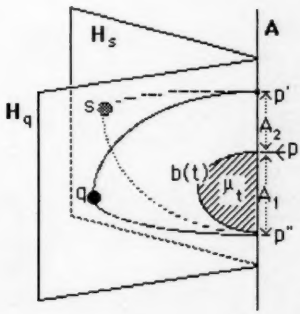

Fig. 17

an exchange, as illustrated in Fig. 16. After the exchange we will obtain a new tiling with the same complexity, however in the new tiling the leaf $b(t_7)$ will be inessential.

The remaining problem is to show that we can actually realize this exchange so that it takes braids to braids and leaves the tiling unchanged, except for reordering of the p_i's. If we can do this, then our exchange move will enable us to reduce the complexity. We need to create a region in 3-space in which we can make the exchange move in a controlled fashion.

Each non-singular $\mathbf{H}_t$ meets $\mathbf{X}$ in a unique arc $b(t)$ which joins p to p' or p'' and cobounds with a part of the axis $\mathbf{A}$ two discs in the fiber $\mathbf{H}_t$. If $b(t)$ joins p to p' (resp. p''), let μ_t be the disc which does not contain p'' (resp. p'). We have sketched in one such arc $b(t)$ and shaded in the disc μ_t in Fig. 17. If we think of the subsurface Δ of Fig. 15 as a boxing glove, the discs μ_t for $t_1 < t < t_5$ will appear to be "outside" the glove, whereas those for $t_5 < t < t_1$ will be "inside" the glove. The disc δ, which is on the glove, is a limiting position for both families of discs. Thus the closure of the union of all of the discs μ_t, $t \in [0, 2\pi]$ will be two 3-balls $\mathbf{B}_1$ and $\mathbf{B}_2$, which intersect along a single arc in the disc δ of Fig. 13, i.e. the arc which runs from q to s through p. The shaded disc in Fig. 14 is in the boundary of $\mathbf{B}_1 \cup \mathbf{B}_2$.

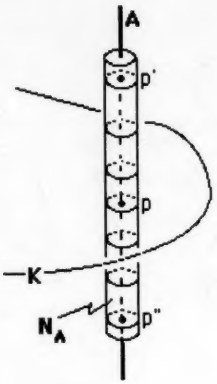

Fig. 18

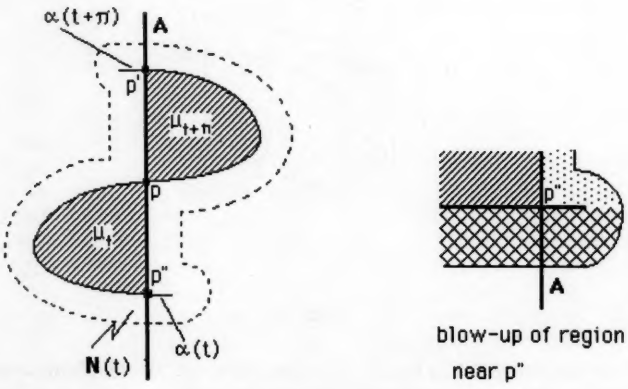

Fig. 19

We may assume that $\mathbf{B}_1 \cup \mathbf{B}_2$ intersects the link $\mathbf{K}$ in two unbraided weighted arcs. This is clearly possible, since all braiding may be pushed out of $\mathbf{B}_1 \cup \mathbf{B}_2$. Notice that the surface $\mathbf{X}$ may meet $\mathbf{B}_1$ and $\mathbf{B}_2$ in some number of sheets which are locally parallel to the embedded disc Δ which we depicted in Fig. 15. In particular, $\mathbf{X}$ may meet a neighborhood $\mathbf{N}_\mathbf{A}$ of $\mathbf{A}$ in some number of radially foliated discs between the discs at p' and p'', as in Fig. 18.

Now we need to thicken $\mathbf{B}_1 \cup \mathbf{B}_2$ a little bit. With this in mind, reparametrize the interval $[0, 2\pi]$ so that the singularities occur at $t_1 = 0$ and $t_5 = \pi$. We can then pair the discs μ_t and $\mu_{t+\pi}$, so that μ_t is in $\mathbf{B}_1$ and $\mu_{t+\pi}$ is in $\mathbf{B}_2$. See Fig. 19, which depicts subsets of $\mathbf{H}_t \cup \mathbf{H}_{t+\pi}$. Now notice (see Fig. 13) that if $\mathbf{H}_t$, $0 < t < \pi$, is non-singular, then $\mathbf{H}_t$ contains two leaves in the foliation of Δ: the leaf $b(t)$, which joins p to p'', and also a small arc $\alpha(t)$ which runs out from p'' to the boundary of Δ.

Similary, $\mu_{t+\pi}$ is a union of $b(t+\pi)$, which joins p to p', and a small arc $\alpha(t+\pi)$ which runs out to the boundary from p''. See Fig. 19. Let $\mathbf{N}(t)$ be a neighborhood of $\mu_t \cup \mu_{t+\pi} \cup \alpha(t) \cup \alpha(t+\pi)$ in $\mathbf{H}_t \cup \mathbf{H}_{t+\pi}$, chosen so that $\mathbf{N}(t) \cap \mathbf{K} = (\mu_t \cup \mu_{t+\pi}) \cap \mathbf{K}$ and $\mathbf{N}(t) \cap \mathbf{X}$ is the union of $(\mu_t \cup \mu_{t+\pi}) \cap \mathbf{X}$ and $(\mathbf{H}_t \cup \mathbf{H}_{t+\pi}) \cap \mathbf{N}_\mathbf{A}$. Finally, choose the neighborhoods $\mathbf{N}(t)$ so that they vary smoothly as t is varied between 0 and π. Let $\mathbf{N}$ be the closure of the union of all of the $\mathbf{N}(t)$, $t \in [0, \pi)$. Notice that $\mathbf{N}$ is not a 3-ball (it is a 3-ball with holes), because $\mathbf{N}(t)$ is not a disc for t-values close to $t = 0$. Chosse $\varepsilon > 0$ so that $\mathbf{N}(t)$ *is* a disc for $t \in [\varepsilon, \pi - \varepsilon]$.

We can now describe precisely the exchange move which was indicated earlier in Fig. 16. See the top left picture in Fig. 20, which is intended to correspond to the left picture in Fig. 19, at some $t \in (\varepsilon, \pi - \varepsilon)$. The plane of the paper is divided by $\mathbf{A}$ into two half-planes, and we will assume the left one to be $\mathbf{H}_t$ and the right one to be $\mathbf{H}_{t+\pi}$. We want to describe an isotopic deformation of $\mathbf{X} \cup \mathbf{K}$, and shall do so by the series of pictures in Fig. 20, all of which correspond to the same fixed value of $t \in [\varepsilon, \pi - \varepsilon]$.

Choose a subdisc $\mathbf{d}$ of $\mu_{t+\pi}$ such that $\mathbf{d} \cap \mathbf{K} = (\mu_{t+\pi}) \cap \mathbf{K}$ and also $\mathbf{d} \cap \mathbf{X} = (\mu_{t+\pi}) \cap \mathbf{X} - b(t+\pi)$. Our isotopy is to be supported in $\mathbf{N}(t)$ and is to be the identity on μ_t and on $\mu_{t+\pi}$ minus a neighborhood of $\mathbf{d}$ on $\mu_{t+\pi}$. The isotopy pushes the disc $\mathbf{d}$ (and the points of $\mathbf{K}$ and $\mathbf{X}$ which meet it) across $\mathbf{A}$ and then down and eventually into the cross hatched area below p'' which was illustrated in the blow-up in Fig. 19. The isotopy is to be defined on pairs of fibers, and is to be defined so that it varies continuously as we vary t. At the end of the isotopy the link $\mathbf{K}$ is to encircle the axis below p'' instead of between p' and p. Also, each sheet of $\mathbf{X}$ which intersected $\mathbf{A}$ between p' and p is to intersect $\mathbf{A}$ the same number of times below p''. (One of the authors likes to think of this change as accomplished by "putting your hand into your pocket and emptying it".)

The only remaining problem is to ask what happens in the t-interval $[0, \varepsilon]$? Let u be the isotopy parameter. The first thing to notice that when u is in the interval $[.25, .75]$ the isotopy is supported in the left half-plane. We may then assume that the deformed discs $\mathbf{d}_u$, $u \in [.25, .75]$, have empty intersection with $\mathbf{H}_t \cup \mathbf{H}_{t+\pi}$ if $t \notin [\varepsilon, \pi - \varepsilon]$. The second thing to notice is that when $u \in [0, .25]$ and $u \in [.75, 1]$ the isotopy is supported in a neighborhood of the axis $\mathbf{A}$. Since that neighborhood can be chosen to be disjoint from a neighborhood of the singular points (see Fig. 14) it follows that Fig. 20 actually tell us everything we need to describe the isotopy of $\mathbf{X} \cup \mathbf{K}$ completely.

At the end of the isotopy we may reposition $\mathbf{X}$ so that the tiling of $\mathbf{X}$ will be exactly as it was before the change. The only change will be in the order of the points of $\mathbf{A} \cap \mathbf{X}$ on $\mathbf{A}$. The interchange of order has an important consequence: one or more essential b-arcs in the foliation of $\mathbf{X}$ will have been changed to inessential b-arcs. We can therefore modify $\mathbf{X}$ by an additional isotopy to a new splitting 2-sphere $\mathbf{X}'$, reducing the complexity. The proof of Lemma 4 is complete. □

We now complete the proof of the split braid theorem. We begin with a splitting 2-sphere $\mathbf{X}$ which has complexity $c(\mathbf{X}, \mathbf{H}) \geqq (2, 0)$. If the complexity is $(2, 0)$ we are done, so assume it is $> (2, 0)$. This implies that $|\mathbf{A} \cap \mathbf{X}| > 2$, because if it were 2, then by our general position hypotheses we would also have $|\mathbf{H} \cdot \mathbf{X}| = 0$. By Lemmas 2 and 3, we can then conclude that there is a vertex of valence 2 in the foliation of $\mathbf{X}$. By Lemma 4 we conclude that after an exchange we may replace $\mathbf{X}$ by a new splitting 2-sphere $\mathbf{X}'$ with smaller complexity. After finitely many such changes we will obtain a splitting 2-sphere $\mathbf{X}''$ which intersects $\mathbf{A}$ twice, and a split n-braid representative $\mathbf{K}'$ of our link $\boldsymbol{K}$. □

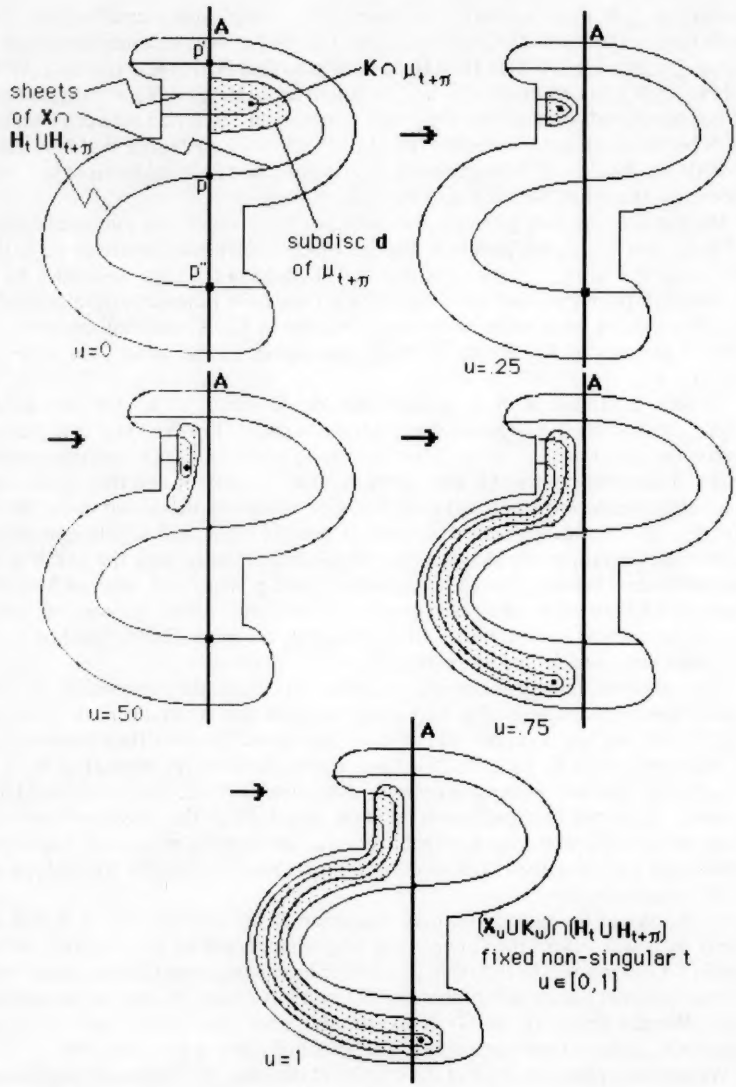

Fig. 20

3. Proof of the composite braid theorem

We begin with a closed n-braid $\mathbf{K}$ which represents a composite link $\boldsymbol{K}$. By the split braid theorem, we know that if $\boldsymbol{K}$ is a split link, we may find splitting 2-spheres which exhibit it as an obviously split closed braid. Thus we may work on one

component at a time. Therefore we may assume that our representative **K** is non-split.

The proof is almost identical with that of the split braid theorem, therefore we will only discuss the points where they differ. We are given a 2-sphere **Y** (replacing **X**) which decomposes **K** into a connected sum of non-trivial link types. The only difference between the 2-sphere **X** of Sect. 2 and our 2-sphere **Y** is that **X** was disjoint from **K**, whereas **K** pierces **Y** twice. Our goal is to modify **K** and **Y** until we achieve a 2-sphere **Y**′ which also realizes the connected sum decomposition and which is pierced twice by the axis **A**.

The general position arguments given in Sect. 2 go through, but a complication occurs in the proof of Lemma 1, when we try to remove SCC's from the foliation of **Y**. The possibility exists that **K** will pierce the discs which are to be used in the surgeries. Suppose that c is a SCC in $\mathbf{H}_t \cap \mathbf{Y}$. Then c bounds a disc δ in $\mathbf{H}_t$, and also divides **Y** into two discs δ_1 and δ_2. Since **K** intersects **Y** twice, it either intersects δ_1 once and δ_2 once, or (by choosing the notation appropriately) it intersects δ_1 twice and misses δ_2. Assume the latter. Then if we surger **Y** along δ, as in the proof of lemma 2, **Y** will be split into two 2-spheres $\mathbf{Y}_1 = \delta \cup \delta_1$ and $\mathbf{Y}_2 = \delta \cup \delta_2$, and $\mathbf{Y}_2$ must be inessential because it misses **K**. Thus we can replace **Y** by $\mathbf{Y}_1$, and proceed as in the proof of Lemma 2. If, on the other hand, **K** meets both δ_1 and δ_2, then both 2-spheres $\mathbf{Y}_1$ and $\mathbf{Y}_2$ will intersect **K** twice, and at least one of them, say $\mathbf{Y}_1$, must be essential. We can then discard $\mathbf{Y}_2$, and proceed as in the proof of Lemma 2, Thus we may assume that **Y** is foliated without SCC's.

Our 2-sphere **Y** admits a tiling. Call a tile *good* if it is not pierced by **K**. Call a vertex of valence 2 or 3 *good* if it is adjacent to a good tile. As in the proof of the split braid theorem, we have to prove there is a vertex of valence 2 or 3 in the tiling, but now we need a little more: we must be sure that the vertex of valence 2 or 3 is good. The equation.

$$2V_2 + V_3 = 8 + V_5 + 2V_6 + 3V_7 + \ldots.$$

still holds. In this equality every entry is non-negative. Now, we can assume (after a small isotopy of **K**) that the two points where **K** pierces **Y** are in the complement of the set of singular leaves, i.e. in a region **R** like the one depicted in Fig. 6. Call **R** a *bad* region if it is pierced by **K**. Now, a region **R** is adjacent to exactly two tile vertices, and since there are at most 2 bad regions this means that there are at most 4 bad vertices. Thus, if $V_2 + V_3 \geqq 5$, we can always find a good vertex, so assume $V_2 + V_3 \leqq 4$. This means that $V_2 \leqq 4$. However, $V_2 + (V_2 + V_3) \geqq 8$. Therefore the only case where we could fail to have a good vertex of valence 2 or 3 is in the very special situation when $2V_2 = 8$ and $V_3 = 0$. This means that **Y** is a union of exactly two tiles, also the complement of the set of singular leaves is a union of four regions, and two of these are pierced by **K**.

We can now proceed, as in the proof of Lemma 4 above, to construct **Y** as an embedded surface in S^3. Let p_1, p_2, p_3, p_4 be the 4 points where **A** pierces **Y**, in their natural cyclic order on **A**. Each of these vertices has valence 2, so there are 2 tiles and 2 singular points in the foliation, say s and q. See the top pictures in Fig. 21. Since all of our vertices are bad, we know that **K** must pierce a pair of opposite regions, say R_1 and R_3. By the first part of the proof of Lemma 4, we know that the signs of s and q are opposite, so the clockwise cyclic order of the tile vertices will be p_1, p_2, p_3, p_4 in, say, the tile which contains s and p_4, p_3, p_2, p_1 in the tile which contains q, when viewed by an upright observer on the positive side of the tile. The points s and q lie in distinct fibers, and the singular leaves will be a union of four

arcs which go out from s(resp. q), in a single fiber, like the spokes in a wheel, ending at the four cyclically ordered vertices, on **A** in a neighborhood **U** of the singular leaves the 2-sphere **Y** will be transverse to each fiber, so there is a unique way to extend the embedding of **S** to an embedding of **U**. With **U** in place, we can attach the four non-singular regions R_1, R_2, R_3, R_4. The embedding of **Y** is then as in the bottom picture of Fig. 21.

By hypothesis each arc in the foliation of **Y** is essential, so our link **K** must be an obstruction to its removal. See Fig. 22. Thus **K** must encircle the axis $n \geqq 1$ times between p_1 and p_2 and also $m \geqq 1$ times between p_3 and p_4, inside **Y**. These strands will in general be braided; we have indicated the braiding as occurring inside two boxes, labeled A and B. Similarly, **K** must encircle the axis $j \geqq 1$ times between p_2 and p_3 and $k \geqq 1$ times between p_4 and p_1, outside **Y**, with braiding occurring in the boxes which are labeled C and D. In addition, one strand of **K** pierces **Y** in the region R_1 and another in the region R_3, to join up the part of **K** which is inside **Y** with the part which is outside **Y**, as illustrated. Thus **K** is a connected sum of a link which is represented as an $(n + m)$-braid and a link which is represented by a $(j + k)$-braid.

It is now clear that we can slide the strands of **K** which pierce **Y** into, say, Region 2. After such a slide we will have a good vertex of valence 2. A complexity-

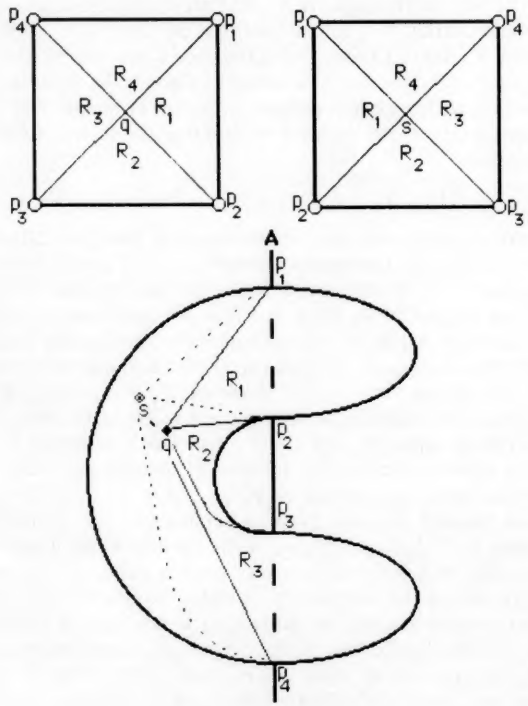

Fig. 21

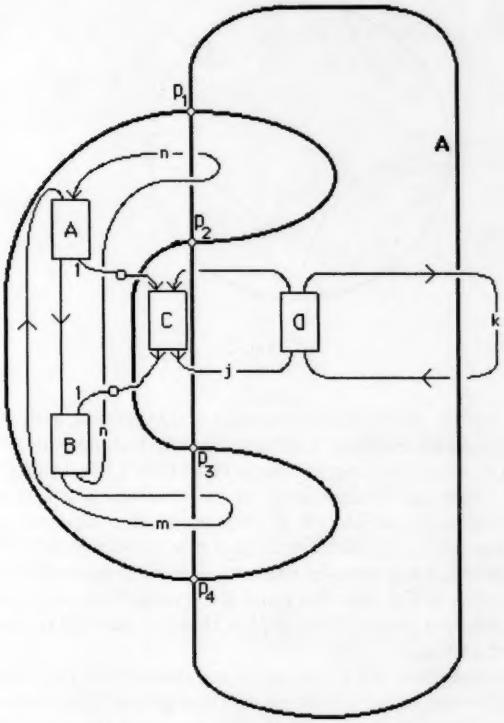

Fig. 22

reducing exchange move is applicable. The proof of the composite braid theorem is complete. □

4. Proof of the 3 and 4-braid theorem

In this section we prove special versions of the split and composite braid theorems which hold when the braid index is 3 or 4. See Sect. 1 above for the statement of the theorems.

Two of the four cases have been settled elsewhere, vis: In [Mu] Murasugi proved, in effect, that every closed 3-braid representative of a split link is conjugate to a split 3-braid representative. The analogous situation for composite links was settled by Morton in [Mo, 1]. A third case, that of split braids of braid index 4, has not been treated elsewhere, however the examples which we gave earlier in Fig. 1 show that there are split links of braid index 4 which are *not* conjugate to split closed braids. Thus the only case which requires proof is the case of composite braids of braid index 4.

Remark. The reader will have no difficulty in supplying new proofs for the three other cases, with the techniques which we will use to settle the fourth.

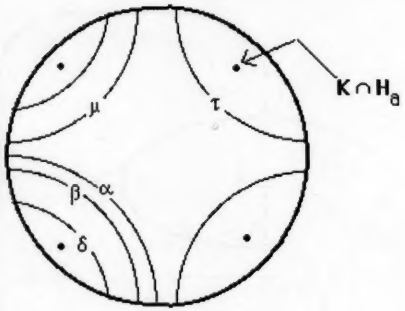

Fig. 23

We begin, as in the proof of the composite braid theorem, with a composite link and a 2-sphere **Y** which realizes the connected sum decomposition. We assume, as usual, that **Y** is in a nice position relative to **H**, so that **Y** has a tiling. Each tile has 4 vertices and 4 edges and 1 singularity in its foliation. The first step will be to examine the components of $\mathbf{H}_t \cap \mathbf{Y}$ as they occur in a sequence of non-singular fibers at times $t = a, b, c, \ldots$. See Fig. 23. Let α be a component of $\mathbf{H}_a \cap \mathbf{Y}$. Then α separates $\mathbf{H}_a$, and since α is essential (see Sect. 2) both components of $\mathbf{H}_a$ split along α must be pierced by **K**. On the other hand **K** is a closed 4-braid, so we know that **K** pierces $\mathbf{H}_a$ in exactly 4 points. This implies that α is parallel to one of the 4 arcs types illustrated in Fig. 23.

We assume that there are n_1, n_2, n_3, n_4 parallel arcs of each type, where each $n_j \geqq 0$. If we follow any one arc as we sweep through the fibration, we know it must ultimately be modified by a surgery. However, arcs in the position of β and δ in Fig. 24 cannot be surgered with one-another because such a surgery would create an inessential arc. Similarly, arcs like α and μ cannot be surgered with one-another, because that too would create an inessential arc. Thus the only possibility is that arcs in the position of α and τ are surgered with one-another. This has the effect of removing α and τ from a pair of opposite groups and adding new arcs α' and τ' to the other pair. This process must continue until there are no more arcs left in one of the pairs, at which point the process must be reversed, because every arc must both be modified by a surgery and also ultimately go back to its initial position. However, this means that eventually α' will be surgered again with τ'. Thus, in fact, if there were more than two arcs our surface **Y** would be disconnected. Hence there are exactly two arcs in $\mathbf{H}_a \cap \mathbf{Y}$, and two singular fibers.

The tiling of **Y** thus contains exactly two tiles, each with four vertices of valence two. This is precisely the situation which was described in the proof of the Composite Braid Theorem in Sect. 3, so we can conclude that the picture is exactly that in Fig. 22, with $n = m = j = k = 1$. Note that this implies immediately that our link $\boldsymbol{K}$ is actually the connected sum of *three*, not just two links, each of which can then only be a type $(2, r_i)$ torus link, for some $r_i \geqq 1$. Thus we have proved that if $\boldsymbol{K}$ is the connected sum of a link of braid index 3 and a link of braid index 2, then **K** is in fact conjugate to a composite 4-braid.

We go back to the case where $\boldsymbol{K}$ has three factors, and construct the 2-sphere which realizes the second connected sum decomposition of $\boldsymbol{K} = \boldsymbol{K}_1 \# \boldsymbol{K}_2 \# \boldsymbol{K}_3$.

Referring momentarily to Fig. 21, let c_1 be an arc on $\mathbf{Y}$ which joins the points s and q in the interior of R_1, and let c_2 be an arc which joins them in the interior of R_3. Then $c = c_1 \cup c_2$ is a simple closed curve which divides $\mathbf{Y}$ into two discs, a disc $\mathbf{Y}_2$ which contains R_2 and a disc $\mathbf{Y}_4$ which contains R_4. (Thus p_2 and p_3 are in $\mathbf{Y}_2$, while p_1 and p_4 are in $\mathbf{Y}_4$). We assume that we chose c_1 and c_2 so that the two points where $\mathbf{K}$ pierces $\mathbf{Y}$ are in $\mathbf{Y}_2$. Then $\mathbf{Y}_4$ is a disc with boundary c which is disjoint from $\mathbf{K}$, and is pierced twice by $\mathbf{A}$, at p_1 and at p_4. Keeping the curve c in mind, we now pass to Fig. 22, where we see that there is another disc, $\mathbf{Z}$ which also has boundary c, which is chosen so that the interior of $\mathbf{Z}$ lies "outside" the 3-ball bounded by $\mathbf{Y}$. In fact, we can choose $\mathbf{Z}$ so that it intersects $\mathbf{K}$ in two points, along the strands which join the blocks labeled C and D, and so that $\mathbf{Z} \cap \mathbf{A}$ is empty. Let $\mathbf{Y}' = \mathbf{Y}_4 \cup \mathbf{Z}$. Then $\mathbf{Y}'$ is also a 2-sphere which realizes the connected sum decomposition of $\mathbf{K}$, and $\mathbf{Y}' \cap \mathbf{A} = 2$ points. Thus, except for the very special case when the block D consist of a single crossing, we see that $\mathbf{K}$ is an obviously composite closed braid.

In the case where D contains a single crossing, our closed braid $\mathbf{K}$ has a trivial loop. That is, $\boldsymbol{K}$ is a connected sum of type $(2, r_1)$ and type $(2, r_2)$ torus links, $r_i \geqq 1$, and it is represented as a closed 4-braid. In this case there actually may be a non-composite 4-braid which represents $\boldsymbol{K}$. The proof is complete. □

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Effectiveness—non effectiveness in semialgebraic and PL geometry

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0. Introduction and statement of the main results

Working in real semialgebraic geometry one could easily have the feeling that the main facts can be realized by algorithms (e.g. path connecting points, triangulations, good stratifications . . .) and the main trouble is to construct reasonably fast algorithms: in fact it is this (in principle) constructive nature of semialgebraic geometry and the existence of powerful computers which make it interesting in view of concrete applications.

On the other hand, Nabutovsky's examples of non recursive functions associated to natural semialgebraic constructions (see [N]) tell us that there are some conceptual limitations to the above attitude.

In this paper we consider some further examples of effectiveness-non effectiveness results in semialgebraic and PL geometry: our starting point was the study of Shiota-Yokoi solution of the subanalytic (hence semialgebraic) *Hauptvermutung* ([SY]) and the attempt to make it effective.

Before starting we need to introduce some notations.

We shall use systematically the notation:

$$[A_1, \ldots, A_h] \to [B_1, \ldots, B_k]$$

to mean that there exists an algorithm accepting the "objects" $A_1, \ldots, A_h$ as input and producing $B_1, \ldots, B_k$ as output.

With the same meaning we shall say also that "$B_1, \ldots, B_k$ *can be effectively constructed, starting from* $A_1, \ldots, A_h$".

Similarly we shall write:

$$[A_1, \ldots, A_h] \nrightarrow [B_1, \ldots, B_k]$$

to mean that "$B_1, \ldots, B_k$ *cannot be effectively constructed starting from* $A_1, \ldots, A_h$", that is there doesn't exist any algorithm producing $B_1, \ldots, B_k$ from $A_1, \ldots, A_h$.

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Some time we shall consider also systems:

$$\begin{cases} [A_1, \dots, A_h] \to [B_1, \dots, B_k] \\ [C_1, \dots, C_s] \to [D_1, \dots, D_r] \end{cases}$$

with the obvious meaning.

A *semialgebraic set* is any subset $X \subseteq \mathbb{R}^n$ (some n) with a *presentation* of the form:

$$X = \bigcup_{i=1}^{h} \bigcap_{j=1}^{k_i} \{f_{ij} *_{ij} 0\}$$

where each $f_{ij} \in \mathbb{R}[X_1, \dots, X_n]$, $*_{ij} \in \{=, >\}$.

For any such presentation set:

$$r = \sum k_i \qquad d = \sup\{\text{degree of } f_{ij}\} .$$

By the notation $X \in \mathscr{A}(n, p, q)$, $n, p, q \in \mathbb{N}$, *we mean that* X *is a semialgebraic set in* $\mathbb{R}^n$ *with a given presentation such that* $r \leqq p, d \leqq q$.

A semialgebraic map $f: X \to Y$ between semialgebraic sets is a continuous map with semialgebraic graph Γ_f.

If K *is a simplicial complex in* $\mathbb{R}^n$ *(some* n*),* $P = |K|$ *is the polyhedron in* $\mathbb{R}^n$ *triangulated by* K; $\# K$ *denotes the number of simplexes in* K. *A PL m-ball is any polyhedron* $B \subseteq \mathbb{R}^n$ *PL isomorphic to the standard m-simplex* Δ_m. *By* K_Δ *we denote the standard triangulation of* Δ_m.

Let us state now some of the results of the paper.

Theorem 1 (strong effective semialgebraic Hauptvermutung for dimension $\leqq 3$; shortly: SEH_m $m \leqq 3$). *Let* K, L *be finite simplicial complexes in* $\mathbb{R}^n$. $\# K, \# L \leqq k$. *Let* $f: |K| \to |L|$ *be a semialgebraic homeomorphism with* $\Gamma_f \in \mathscr{A}(n^2, p, q)$. *Assume* $m = \dim |K| = \dim |L| \leqq 3$. *Then*

$$\begin{cases} [K, L, f] \to [g: K' \to L'] \\ [n, k, p, q] \to [D] \end{cases}$$

where K' *is a simplicial subdivision of* K, L' *of* L, g *is a simplicial isomorphism and* $\# K', \# L' \leqq D \in \mathbb{N}$ *(see Theorem* 3.5).

Theorem 2 (weak effective semialgebraic Hauptvermutung for any dimension; shortly: WEH_m $\forall m$). *With the same notation as in Theorem* 1 *but with arbitrary* $m = \dim|K| = \dim|L|$, *one has*

$$[K, L, f] \to [g: K' \to L']$$

(*see* 3.3).

In the PL setting we have the following *effective trivialization of PL ball*:

Theorem 3. *Let* $B = |K|$ *be a triangulated PL m-ball,* $|K| \subseteq \mathbb{R}^m$, $\# K \leqq d$, $m \leqq 3$. *Then*

$$\begin{cases} [K] \to [g: K' \to L] \\ [d] \to [D] \end{cases}$$

where K' *is a simplicial subdivision of* K, L *is a subdivision of the standard triangulation of* Δ_m, g *is a simplicial isomorphism,* $D \in \mathbb{N}$ *and* $\# K' \leqq D$ (*see Problem* 2.2 *and Proposition* 2.23 *and* 2.24).

Remark. If $m \leqq 2$ we are able to omit the hypothesis $|K| \subseteq \mathbb{R}^m$, that is we may assume $|K| \subseteq \mathbb{R}^n$, $n \neq m$ in general.

Theorem 4. *With the same notation as in Theorem 3, if m is arbitrary and* $|K| \subseteq \mathbb{R}^n$, *with* $m \neq n$ *in general, then*

$$[K] \to [g: K' \to L]$$

(*see Problem* 2.1 *and Corollary* 2.17).

Remark. Theorems 3 and 4 are, in some sense, PL counterparts of Theorems 1 and 2; in fact they shall be employed to prove the first ones.

The SEH_m for $m \geqq 4$ is still open (to us). *However the "PL counterpart" is false,* that is we have the following *non effectiveness* result for trivialization of PL-balls of higher dimensions.

Theorem 5. (*a*) *For every* $m, n, d \in \mathbb{N}$ *there exists* $s = s(m, n, d) \in \mathbb{N}$ *such that for every PL m-ball* $B = |K| \subseteq \mathbb{R}^n$, *triangulated by K with* $\# K \leqq d$, *there exists a simplicial isomorphism* $g: K' \to L$ *where* K' *is a subdivision of K, L of* K_Δ, *and* $\# K' \leqq s$.

(*b*) *For every* $m \geqq 6$

$$[m, d] \mapsto [D]$$

such that $D \in \mathbb{N}$ *and* $s(m, d) = s(m, m, d) \leqq D$ (*i.e.* $s(m, d)$ *cannot be bounded by any recursive function of* m, d) (*see Corollary* 2.18 *and Remark* 2.21).

In the paper we are concerned with the opposition 'effectiveness-non effectiveness"; we shall not consider the problem of the complexity of the algorithms (when they exist).

We shall assume the existential solution of the (semialgebraic) Hauptvermutung by Shiota-Yokoi; however the eventual effectiveness (strong or weak) is not a consequence of their proof, not even in low dimensions: we shall discuss shortly the reason of it in (3.10).

For basic facts about semialgebraic geometry we refer to [BCR] or to [BR] (in the second one some effectiveness questions are explicitly pointed out).

For basic facts about PL-geometry we refer to [RS] and [Hu].

1. Recall on a result of Novikov

Integrating Smale's *h*-cobordism theorem (and its consequences) with the results of Adyan, Markov, on the "non-decidability of the triviality for finitely presented groups", Novikov (see [VKF]) proved the following result (in a formulation which is more convenient to our aim).

Theorem (Novikov). *For every* $m \geqq 6$ *one can construct a sequence* $\mathcal{M} = \{M_S = |K_S|\}_{S \in \mathbb{N}}$ *of triangulated PL submanifolds of* $\mathbb{R}^m$, $\dim M_S = m$, $\partial M_S \neq \varnothing$ *such that*

(*i*) $$[|K_S| \in \mathcal{M}] \mapsto [t]$$

where $t \in \{$*yes, no*$\}$ *and t is "yes" if* M_S *is a PL m-ball, t is "no" otherwise.*

(*ii*) $$[|K_S| \in \mathcal{M}] \mapsto [t]$$

where $t \in \{yes, no\}$ and t is "yes" if ∂M_S is a PL $(m-1)$-sphere, i.e. PL isomorphic to $\partial\Delta_m$), "no" otherwise.

Novikov's theorem shall be one of the main tools in order to prove Theorem 5 of the Introduction.

The idea to use it (in fact a smooth analogous of it) in order to produce examples of non recursive functions in semialgebraic geometry is due to Nabutovsky (see [N]).

2. On the effective trivialization of PL-balls

We shall adopt the notation of [RS], in particular: we distinguish between m-cells and *PL* m-balls; if A, B are cells $A \leqq B$ means that A is a *face* of B; if K and L are (cell or simplicial) complexes, $L \lhd K$ means that L is a *subdivision* of K; $\mathrm{St}(A, K)$ denotes the *star* of A in K.

By definition a PL m-ball is PL isomorphic to the standard m-simplex Δ_m. We denote by K_Δ the standard triangulation of $\Delta = \Delta_m$.

By a *trivialization* of B we mean a simplicial isomorphism $g: K \to L$ where $B = |K|$ and $L \lhd K_\Delta$.

Problem 2.1 (WT_m: weak effective trivialization problem in $\dim m$). *It asks for*

$$[K] \to [g: K' \to L]$$

where $B = |K|$ is any triangulated PL m-ball, $K' \lhd K$, $L \lhd K_\Delta$, g is a simplicial isomorphism.

Problem 2.2 (ST_m: strong effective trivialization problem in $\dim m$). *It asks for*

$$\begin{cases} [K] \to [g: K' \to L] \\ [d] \to [D] \end{cases}$$

where $B = |K|$ is any triangulated PL m-ball, with $\# K \leqq d$, $K' \lhd K$, $L \lhd K_\Delta$, g is a simplicial isomorphism $D \in \mathbb{N}$ and $\# K' \leqq D$.

We shall see, in particular, that WT_m has solution for every m; on the contrary ST_m has not solution for $m \geqq 5$. In low dimension we have some stronger results.

(*A*) *Recall of some simple facts of PL-geometry*

We shall use systematically the following well-known facts which we state with emphasis on effectiveness.

Lemma 2.3. *For any cell-complex K with a number of cells $\# K \leqq d$ one has*

$$\begin{cases} [K] \to [H] \\ [d] \to [D] \end{cases}$$

where H is a simplicial complex such that $H \lhd K$, $\# H \leqq D \in \mathbb{N}$, the 0-skeleton H^0 and K^0 coincide.

Proof. See [RS], Proposition 2.9. □

Lemma 2.4. *If K, M are two simplicial complexes such that $|K| = |M|$, $\#K$, $\#M \leqq d$ then one has:*

$$\begin{cases} [K, M] \to [H] \\ [d] \to [D] \end{cases}$$

where H is a simplicial complex such that $H \lhd K$, M, $\#H \leqq D \in \mathbb{N}$.

Proof. Apply (2.3) to the cell complex $K \cap M$. □

Two cell-complexes K, L are said *formally isomorphic* if there exists a bijection $q: K \to L$ such that $\forall \sigma, \tau \in K\ \sigma \leqq \tau \Rightarrow q(\sigma) \leqq q(\tau)$.

Lemma 2.5. *If K, L are cell-complexes, $q: K \to L$ is a formal isomorphism, $\#K$, $\#L \leqq d$ then one has:*

$$\begin{cases} [q: K \to L] \to [f: H \to M] \\ [d] \to [D] \end{cases}$$

where H, M are simplicial complexes, $H \lhd K$, $M \lhd L$, f is a simplicial isomorphism, $\#H = \#M \leqq D \in \mathbb{N}$.

Proof. The construction of H in Lemma 2.3 uses a suitable order on the cells of K. Induce the order on L via q. Construct M as in 2.3 using this order. One gets two formally isomorphic simplicial complexes, hence simplicially isomorphic. □

(*B*) *Effective shelling and effective trivialization of PL balls*

Let $B = |K|$ be a triangulated PL m-ball.

Definition 2.6. A shelling of K is an ordering of the m-simplexes of K:

$$\sigma_1, \ldots, \sigma_h$$

such that for every $i = 1, \ldots, h-1$

$$B_i := \overline{B \setminus \bigcup_{1 \leqq j \leqq i} \sigma_j} \qquad (B_0 = B)$$

is a PL m-ball (triangulated by the restriction of K).

Remark 2.7. If $m \leqq 2$ every triangulation K of a PL m-ball B carries some shelling (for $m = 1$ is trivial, for $m = 2$ see [Mo]).

On the contrary, if $m \geqq 3$ there are triangulations of PL m-balls (in fact of Δ_m) without any shelling (see [R]). It is known (see [BM; S], and also [Ac]) that for any triangulation of any PL m-ball there exists a subdivision admitting some shelling.

The proof (see [BM]) in the general case is not effective and uses the existence of a PL isomorphism with Δ_m. In fact we want to prove that the effectiveness of shellings (up to subdivision) is *equivalent* to the effectiveness of trivializations.

In analogy with the trivialization problems we can state:

Problem 2.8 (WS_m: weak effective shelling problem in dim m). *It asks for*

$$[K] \to [K', \mathscr{S}]$$

where $B = |K|$ is any triangulated PL m-ball, $K' \lhd K$, and $\mathscr{S}$ is a shelling of K'.

Problem 2.9 (SS_m: strong effective shelling problem in dim m). *It asks for*

$$\begin{cases}[K] \to [K', \mathscr{S}] \\ [d] \to [D]\end{cases}$$

where $B = |K|$, $K' \lhd K$, $\mathscr{S}$ *are as above,* $\# K \leqq d$, $D \in \mathbb{N}$ *and* $\# K' \leqq D$.

We want to prove:

Proposition 2.10. *For every m, the strong (weak) trivialization problem for PL m-balls has solutions if and only if the strong (weak) shelling problem has solutions. Shortly:* $WT_m \Leftrightarrow WS_m$ *and* $ST_m \Leftrightarrow SS_m$.

Proof. We shall prove $ST_m \Leftrightarrow SS_m$. The proof works also in the weak case.

(*a*) $ST_m \Rightarrow SS_m$

It is enough to find solutions of SS_m for any $K \lhd K_\Delta$ with $\# K \leqq d$. This is a particular case of a more general statement.

Lemma 2.11. *If* $C = |K|$ *is any triangulated m-cell in* $\mathbb{R}^m$ *(i.e. C is a convex m-ball),* $\# K \leqq d$ *then one has*

$$\begin{cases}[K] \to [K', \mathscr{S}] \\ [d] \to [D]\end{cases}$$

where $K' \lhd K$, $\# K' \leqq D \in \mathbb{N}$, *and* $\mathscr{S}$ *is a shelling of* K'.

We postpone the proof.

(*b*) $ST_m \Leftarrow SS_m$

Assume, for the moment, the following lemma:

Lemma 2.12 (effective trivialization of collars of $\partial \Delta_m$). *If* $L \lhd K_\Delta$, $\# L \leqq p$ *then one has*

$$\begin{cases}[L] \to [L', h\colon C \to M] \\ [p] \to [q]\end{cases}$$

where $L' \lhd L$, $\# L' \leqq q \in \mathbb{N}$, $|M| = \partial \Delta_m \times [0, 1]$, *C is a subcomplex of L' triangulating a collar of* $\partial \Delta_m$ *and* $h\colon C \to M$, *is an effective trivialization such that* $h_{|\partial \Delta_m} = id \times \{0\}$.

We shall prove Lemma 2.12 later. It is enough to find solutions of ST_m for any triangulation K of $B = |K|$, with $\# K \leqq d$ and a given shelling $\sigma_1, \sigma_2, \ldots, \sigma_r$, where r is the number of the m-simplexes of K. We do it by induction on r. For $r = 1$ it is trivial. Assume that we have done for $(r-1)$ and try to do for r. Set $|K_1| = \overline{K \setminus \sigma_1}$: it is (by hypothesis) a PL m-ball triangulated by the restriction of K, say K_1. $\mathscr{S}$ restricts to a shelling $\mathscr{S}_1 = (\sigma_2, \ldots, \sigma_r)$ and K_1 has $(r-1)$ m-simplexes; $\# K_1 \leqq d$.

By induction we have a solution of ST_m for K_1, that is we construct effectively $K_1' \lhd K_1$, $L_1 \lhd K_\Delta$, $g_1\colon K_1' \to L_1$ a simplicial isomorphism, $D_1 \in \mathbb{N}$ (depending on d) such that $\# K_1' \leqq D_1$.

By pulling-back via g_1 a collar of $\partial \Delta_m$ in Δ_m given by Lemma 2.12 we may assume also that there exists a subcomplex T of K_1' such that $|T|$ is an effectively trivialized collar of $\partial |K_1'|$ in $|K_1'|$, by $h\colon T \to M$ where $|M| = \partial |K_1'| \times [0, 1]$, $h_{|\partial |K_1'|}$:

$\partial|K_1'| \to \partial|K_1'| \times \{0\}$ is $id \times \{0\}$. We want to construct a simplicial isomorphism

$$f: \hat{K} \to K_1''$$

where $\hat{K} \lhd K$, $K_1'' \lhd K_1'$ and $\#\hat{K}$ is effectively controlled by d. Thus we shall complete the proof by taking the simplicial map $g = g_1 \circ f: |K| \to \Delta_m$.

σ_1 intersect $|K_1|$ along $\partial|K_1|$ and the union of the $(m-1)$-faces of K_1 in $\sigma_1 \cap \partial|K_1|$ is an $(m-1)$-ball $B' = \sigma_1 \cap \partial|K_1|$. Using the trivialized collar we have that $B'' = h^{-1}(B' \times [0,1])$ is isomorphic to $B' \times [0,1]$ with an effectively given simplicial isomorphism (for a suitable subdivision). To conclude it is enough to show that $id: \overline{\partial B'' \backslash B'} \to \overline{\partial B'' \backslash B'}$ extends to a PL isomorphism $q: \sigma_1 \bigcup_{B'} B'' \to B''$ which becomes simplicial for suitable subdivisions effectively constructed with bounds on the number of simplexes.

Set $B''' = \sigma_1 \bigcup_{B'} B''$. First remark that the complement in $\partial B'''$ of the interior of $\partial B'' \backslash B'$ coincides with the complement in $\partial\sigma_1$ of the interior of B' and also that $\partial B'' = B' \cup \overline{\partial B'' \backslash B'}$.

Both B' and $\overline{\partial\sigma_1 \backslash B'}$ are cones over their intersection $B' \cap \overline{\partial\sigma_1 \backslash B'}$, hence we can extend q to an isomorphism $q': \partial B''' \to \partial B''$.

Finally, using again the effective trivialization of the collar, we can realize both B''' and B'' as cones over their boundary; hence we can extend to $q'': B''' \to B''$. Remark that this is the proof of 3.25 of [RS], which becomes effective because of the effective trivialization of the collar. The proposition is proved (assuming Lemma 2.12). □

Remark 2.13. The same proof shows also that ST_m and SS_m are equivalent to the following

$$\begin{cases} [K] \to [K', \mathscr{S}] \\ [d] \to [D] \end{cases}$$

where $B = |K|$, is any triangulated PL m-ball, $|K'| = B$ and $\mathscr{S}$ is a shelling of K'; $\#K \leqq d$, $\#K' \leqq D \in \mathbb{N}$ (that is we do not require that $K' \lhd K$). Similarly for the weak statements.

Proof of Lemma 2.11. With a slightly different statement the proof is contained in [Ac]. For the sake of completeness we sketch the construction with emphasis on effectiveness. The proof is by induction on n.

Let B be a convex triangulated n-polyhedron, $B = |K|$. We shall describe an explicit subdivision which is shellable.

Consider a family $E_0, \ldots, E_S$ of parallel hyperplanes with the following properties:

i) $E_0 \cap K = \{v^0\}$; v^0 is a vertex.
ii) each vertex v of K is in some E_j.
iii) In a suitable coordinate system

$$E_j = \{x_n = \alpha_j\} \quad \text{with} \quad \alpha_0 < \alpha_1 < \ldots < \alpha_S$$

It is easy to see that one can do this effectively.

$E_j \cap B$ is a convex $(n-1)$-polyhedron (or possibly a single vertex). By Lemma 2.4 and the induction hypothesis we can subdivide explicitly the cellularization induced by K on $E_j \cap B$ in a shellable triangulation K_j'. Let

$$F_i = B \cap \{\alpha_{i-1} \leqq x_n \leqq \alpha_i\} \quad i = 1, \ldots, s$$

F_i is a convex polyhedron with a cellularization induced by K, K'_{i-1} and K'_i. We can find a triangulation of F_i, $i = 1, \ldots, s$ without introducing new vertices. So we obtain a linear simplicial subdivision K' of K such that the vertices of K' lie in $E_0 \cup \ldots \cup E_S$. Define:

$$V_{i,k} = \{\sigma \in K' | \dim \sigma = m, \sigma \subset F_i, \dim(\sigma \cap E_{i-1}) = k - 1\} .$$

If $\sigma \in V_{i,k}$ and $\sigma' \subset F_i$ is such that σ and σ' have a common $(m-1)$-face, then $\sigma' \in V_{i,k-1} \cup V_{i,k} \cup V_{i,k+1}$. So we can define the *worst* type simplices in $V_{i,k}$ as the simplices which have no common $(m-1)$-face with any simplex in $V_{i,k+1}$.

It is rather easy to prove, as in [Ac], that the following rules define a shelling $\mathscr{S}$ for K'.

(*1*) decreasing i.
(*2*) increasing k.
(*3*) for fixed i and k, first the worst type simplices with more free faces.

So the lemma is proved. □

Proof of Lemma 2.12. By means of Lemma 2.4 it is enough to construct an effectively trivialized collar of $\partial \Delta_m$ for some triangulation of Δ_m. Consider Δ_m as the cone over $\partial \Delta_m$ with center in the barycenter a of Δ_m.

Every $v \in \Delta_m$ is uniquely expressed as $v = ta + (1-t)v'$ where $t \in [0, 1]$ and $v' \in \partial \Delta_m$. Set:

$$\Delta(s) = \{v \in \Delta_m | v = sa + (1-s)v' \quad \text{for some } v' \in \partial \Delta_m\} \quad 0 \leqq s \leqq 1$$

Note that $\Delta(0) = \partial \Delta_m$. Consider $\Delta(\frac{1}{2}) \subseteq \Delta_m$. Using the cone structure we find a natural cell-decomposition of Δ_m having as vertices the union of the vertices of $\Delta(0)$ and $\Delta(\frac{1}{2})$ and such that the induced cell-decomposition on $\bigcup_{0 \leqq s \leqq \frac{1}{2}} \Delta(s)$ is formally isomorphic to the standard cellular decomposition of $\partial \Delta_m \times I$ (i.e. the product $K_{\Delta | \partial \Delta_m} \times I$). Thus we can conclude using Lemma 2.4 and Lemma 2.5. □

Remarks 2.14. (*i*) The proof of 2.13 works for any m-cell in $\mathbb{R}^n$.
(*ii*) Consider $K = K_\Delta$ and let $K^{(r)}$ denote the rth-barycentric subdivision of K. One can explicitly construct (it is elementary but not completely trivial) a simplicial isomorphism between the subcomplex of $K^{(2)}$, $\mathrm{St}(\partial \Delta_m, K^{(2)})$ with a suitable subdivision of $\partial \Delta_m \times I$, (obtaining in this way another proof of 2.12): PL topology tells us (see [RS]) that if $M = |K|$ is a triangulated manifold with boundary, $\mathrm{St}(\partial M, K^{(2)})$ is a collar of ∂M in M. The problem to us is to make the trivialization effective. We have not been able to do it in general, not even under the hypothesis that M is a PL ball (but K is arbitrary); in fact there are some evidence that it should not be possible in higher dimensions see (Remark 2.19 and Corollary 2.20).

(*C*) *Listing bounded subdivisions and effective trivialization of PL balls*

The aim of this section is to prove the following:

Proposition 2.15 (listing bounded subdivisions). *Let $X = |K|$ be any triangulated polyhedron in $\mathbb{R}^n$ with $\# K \leqq d$ and fix a positive integer $D \geqq d$ then we have*

$$\begin{cases} [X = |K|] \to [\{H_1, \ldots, H_s\}] \\ [d, n, D] \to [S] \end{cases}$$

where $S \in \mathbb{N}$ and $s \leqq S$, for every $i = 1, \ldots, s$, $H_i \lhd K$, $\# H_i \leqq D$, and for every $H \lhd K$ such that $\# H \leqq D$ there exists i and a simplicial isomorphism ϕ: $H \to H_i$.

Before proving the proposition we deduce some corollaries. In particular we prove Theorem 4 and Theorem 5 of the Introduction.

Lemma 2.16. *Given $X = |K|$, $Y = |H|$ triangulated polyhedra, we can effectively decide if there exists a simplicial isomorphism $f: K \to H$.*

Proof. In the worse case (that is if there are not evident distinctions like $\dim X \neq \dim Y$ or $\# K \neq \# H$) we have to look for formal isomorphism between K and H, thus we have to analyze a finite number of possibilities. □

Corollary 2.17 (solution of the weak trivialization and shelling problems). *For every m WT_m and WS_m have solution.*

Proof. Since $\mathrm{WT}_m \Leftrightarrow \mathrm{WS}_m$ (Proposition 2.10) it is enough to show that WT_m (for example) has solutions. Take a triangulated m-ball $B = |K|$. By hypothesis there exists some $K' \lhd K$, $\# K' = R_0$, $L \lhd K_\Delta$ such that $g: K' \to L$ is a simplicial isomorphism. If $r = \# K$, for every $R \geqq r$ we can list (Proposition 2.15) all the subdivisions of K with no more than R simplexes and all the analogous subdivisions of K_Δ. Using Lemma 2.16, for every R we can check if there are any such subdivisions which are isomorphic. By hypothesis there exists some R_0 such that the algorithm stops. □

Corollary 2.18 (non effectiveness for the strong trivialization and shelling problems). *For $m \geqq 5$ ST_m and SS_m do not have solution. More precisely: if $m \geqq 6$ they do not have solution even under the stronger hypothesis that the m-balls are embedded in $\mathbb{R}^m$; if $m = 5$, ST_5 and SS_5 fail even for codimension one 5-balls in $\mathbb{R}^6$.*

Proof. Assume first $m \geqq 6$ and ST_m has a solution. Take any triangulated manifold $M_S = |K_S| \subseteq \mathbb{R}^n$ given by Novikov theorem recalled in Sect. 1. If $\# K_S \leqq d$ and if M_S should be a PL m-ball one could effectively determine $D \in \mathbb{N}$ such that there exists a simplicial isomorphism $g: K'_S \to L_S$ where $K'_S \lhd K_S$ and $\# K'_S \leqq D$, $L_S \lhd K_\Delta$. Using the listing of bounded subdivisions as in the previous proof one could decide if M_S is isomorphic to Δ_m (by actually finding isomorphic subdivisions with no more than D simplexes or concluding that they are not PL isomorphic otherwise). This contradicts Novikov's theorem.

Let be $m = 5$ and suppose that ST_5 has a solution. Take again $M_S = |K_S| \subseteq \mathbb{R}^6$, take off a simplex σ from the boundary ∂M_S and consider $N_S = \partial M_S \backslash \sigma$. N_S is a triangulated 5-manifold and by ST_5 and Lemma 2.16 we can decide whether N_S is a 5-ball or not. So we can decide whether ∂M_S is a 5-sphere or not and finally, by h-cobordism in dimension 5, we can decide whether M_S is a 6-ball or not, contradicting Novikov's theorem. □

Remark 2.19. As a consequence of Corollary 2.18 we cannot have a strongly effective proof of h-cobordism theorem: indeed, if so, taking off a simplex τ from the interior of a triangulated m-ball B, one could explicitly triangulate $\overline{B \backslash \tau}$ as the product $\partial B \times I$; this would be enough to have ST_m. A rough analysis of the proof of h-cobordism theorem shows that there are two a priori non effective steps: the first is the trivialization of the handle decomposition, where trivialization of collars is an essential tool; the second is the choice of a disk bounded by the curve of "Whitney trick".

Actually 2.18 has also a rather surprising corollary: collars cannot be strongly effectively trivialized.

We thank A. Marin who remarked it. The argument is the following.

Recall that, if not specified otherwise, by PL m-balls we mean balls generally embedded in some $\mathbb{R}^n$, n and m not necessarily equal. Given a PL m-ball B with boundary S, a *trivialized collar* of S in B is a PL homeomorphism between a regular neighbourhood N of S in B and $S \times [0, 1]$, identifying identically S and $S \times \{0\}$. We have, with the obvious meaning, a *strong effectiveness problem for construction of such trivialized collars.* In case of positive answer, we say shortly that *collars can be trivialized.*

Corollary 2.20 (non effectiveness for the trivialization of the collars). *There exists m, $3 < m < 7$, such that collars in m-balls cannot be trivialized.*

Proof. Let m_0 the minimal number m such that ST_m (extended to all balls) has negative answer. By Corollary 2.18 m_0 exists and in fact $2 < m_0 < 6$. Then one deduces that $(m_0 - 1)$-spheres can be strongly effectively trivialized on the boundary of the $(m_0 - 1)$-simplex (otherwise we contradict the minimality of m_0). For every m_0-ball B consider the $(m_0 + 1)$-ball B' obtained by coning over B. If collars of the boundary S' of B' can be strongly effectively trivialized, the above remark on the trivialization of the boundary of B together with the *unicity of links* (which is a strongly effective result) imply that B can be strongly effectively trivialized. This is a contradiction. □

Remark 2.21. Since Novikov's manifolds M_S are of codimension 0 in $\mathbb{R}^m$, we have proved the point (*b*) of Theorem 5 stated in the Introduction. It remains to prove the point (*a*). It is a consequence of a more general fact:

Lemma 2.22. *For every $n, d \in \mathbb{N}$ there exists $D = h(n, d)$ such that for every couple of triangulated polyhedra $X = |K|$, $Y = |H| \subseteq \mathbb{R}^n$ with $\# K, \# H \leqq d$, if X, Y are PL isomorphic then there exist subdivisions $K' \lhd K$, $H' \lhd H$ with $\# K', \# H' \leqq D$ which are simplicially isomorphic.*

Proof. It is a consequence of (the proof of) the theorem of local triviality of semialgebraic maps as stated and proved, for example, in [BCR] Theorem 9.3.1, following the proof of Hardt ([Ha]) (see also [BR], 2.8.3.). Note that in our case the situation is much simpler because we are working with PL objects.

Every triangulated polyhedron in $\mathbb{R}^n$ with no more than d simplexes can be presented using no more than $\bar{d}$ (effectively computed by means of n and d) polynomial of degree 1, is such a way that it is given as the union of its simplexes.

We can identify the polynomial in n variables of degree 1 with $\mathbb{R}^{n+1}$ and set for every $h \leqq \bar{d}$

$$\mathbf{R} = \mathbf{R}_h = (\mathbb{R}^{n+1})^h$$

Set:

$$S = S_h = \{ >, =, < \}^h$$

Take $\mathbb{R}^n \times \mathbf{R}$ with the natural projections

$$\pi\colon \mathbb{R}^n \times \mathbf{R} \to \mathbf{R} \qquad q\colon \mathbb{R}^n \times \mathbf{R} \to \mathbb{R}^n$$

For every $t = (t_1, \ldots, t_h) \in S$ consider

$$V_t = \{(x, P_1, \ldots, P_h) \in \mathbb{R}^n \times \mathbf{R}\colon P_i(x) t_i 0\}$$

Finally set for every k:

$$\mathcal{V} = \mathcal{V}_{h,k} = \{V \mid V = V_{t_1} \cup \ldots \cup V_{t_k} \quad t_i \in S \quad 1 \leqq i \leqq k\}$$

$\mathscr{V}$ is a finite set. It is easy to find an effective upper bound B for k (in term of $\bar{d}$) such that every triangulated polyhedron with no more than d simplexes is of the form $X = q(\pi^{-1}(v^0) \cap V)$ for some $v^0 \in \mathbf{R}$, $V \in \mathscr{V}$, $h, k \leqq B$. Apply the local triviality theorem to the maps $\pi_{|V}: V \to \mathbf{R}$, V varying in $\mathscr{V}$. It turns out from the proof that $\pi_{|V}$ is trivialized over a finite number of semialgebraic subsets of $\mathbf{R}$ giving a partition of $\mathbf{R}$ (in fact in our case they are defined by polynomial of degree 1). More over we note that:

(*1*) For every V the number of such subsets of $\mathbf{R}$ is effectively bounded in term of n and $\bar{d}$ (hence of d).
(*2*) $\# \mathscr{V}$ is effectively bounded.
(*3*) Two fibers of the same trivialization are PL isomorphic by a simplicial isomorphism between subdivisions with bounded number of simplexes (see [BR] where all this facts are explicitly noted).

Eventually we may have fibers of different trivializations which are nevertheless PL isomorphic. Anyway there is a finite number of possibilities hence the function $D = h(n, d)$ exists. □

Proof of Proposition 2.15. It is again an application of the proof of the local triviality for semialgebraic maps ([Ha; BCR; BR]) adapted (and simplified) in our setting. Given a subdivision $K' \lhd K$, $X = |K|$, $\# K' \leqq D$ then the polyhedron $X = |K| \subseteq \mathbb{R}^n$ is presented as union of the simplexes of K' using no more than $\bar{D}$ polynomials of degree 1, and $\bar{D}$ is effectively computable using n, d and D. Let $P_1, \ldots, P_r$ be such polynomials. They produce a cell-decomposition of X by cells of the form:

$$\sigma_t = \{x \in \sigma \mid P_i(x) t_i 0\}$$

where σ is a simplex of K, $t = (t_1, \ldots, t_r)$ and t_i is a value in $\{>, =, <\}$ associated to P_i, $i = 1, \ldots, r$.

It is clear that such a cell-decomposition subdivides K and has a number of cells effectively bounded.

On the other hand every subdivision $K' \leqq K$ with $\# K' \leqq D$ can be obtained by some cell decomposition of the previous type (that is defined by no more than $\bar{D}$ polynomials of degree 1) by deleting some cells: since the number of the cells is effectively bounded, then there is only a finite number of possibilities to delete cells in order to get (if any) a required subdivision of K.

We work similarly to the proof of Lemma 2.22.

Let $A \subseteq \mathbb{R}^{n+1}$ be the subset of polynomials of degree 1 with zero set intersecting X. Set for $h \leqq \bar{D}$:

$$\mathbf{R} = \mathbf{R}_h = A^h \qquad S = S_h = \{>, =, <\}^h$$

$$\pi: \mathbb{R}^n \times \mathbf{R} \to \mathbf{R} \qquad q: \mathbb{R}^n \times \mathbf{R} \to \mathbb{R}^n$$

For every $t = (t_1, \ldots, t_h) \in S$ set

$$V_t = \{(x, P_1, \ldots, P_h) \in X \times \mathbf{R}: P_i(x) t_i 0\}$$

$$\mathscr{V} = \mathscr{V}_{h,k} = \{V \mid V = V_{t_1} \cup \ldots \cup V_{t_k}\ t_i \in S\ 1 \leqq i \leqq k\}$$

$k \leqq B$ effectively bounded

Apply the local trivialization theorem to $\pi_{|V}: V \to \mathbf{R}$, V varying in $\mathscr{V}$. For every $r \in \mathbf{R}$ $\pi_{|V}^{-1}(r)$ is a cell-subdivision of X of the type described above.

It turns out from the proof of the triviality theorem that, in particular, two fibers in the same trivialization of π over some $\mathbf{R}_i$ of the partition of $\mathbf{R}$ are formally isomorphic; hence the subdivision of K obtained by deleting faces with the same name in both such fibers are simplicially isomorphic. Since the number of $\mathbf{R}_i$ is effectively bounded the proposition is proved. Note that in this way we may obtain several representatives of simplicial isomorphism class of subdivisions of K presented by no more than $\bar{D}$ polynomials; what is important is that we get at least one representative for each class. □

(*D*) *Strong effective trivialization of PL balls in low dimension*

In this section we shall prove, in particular, Theorem 3 of the Introduction.

Proposition 2.23. SS_m *and* ST_m *have solution for* $m \leqq 2$.

Proof. Since $SS_m \Leftrightarrow ST_m$ (2.11) it is enough to show SS_m (for example). Now as already noted in 2.8 *every* triangulation of a PL m-ball with $m \leqq 2$ carries some shelling and at least one of these can be detected effectively. □

Proposition 2.24. SS_3 *and* ST_3 *have solutions for PL 3-balls embedded in* $\mathbb{R}^3$.

Proof. ST_3 is a consequence of an effective version of the proof of the PL Shönflies theorem in $\mathbb{R}^3$; we follow Moise's statement (see Theorem 17.12 in [Mo]), whose proof is very similar to the original one (see [A1]). Since what follows Lemma 1, p. 122 of [Mo] is straightforwardly effective, it is enough to prove the effectiveness of the first part. Here are the steps.

(*1*) *The effective push property EPP*

Let C be a polyhedral 3-cell in $\mathbb{R}^3$ and $D_1 \subset \partial C^3$ be a polyhedral 2-cell. Let J be the boundary of D_1 and define $D_2 = \overline{\partial C \backslash D_1}$. *We say that* C *has the EPP at* D_1 *if for each closed neighborhood* N *of* $C \backslash J$ *one has*:

$$\begin{cases} [K, G, H] \to [f\colon H' \to L] \\ [d] \to [D] \end{cases}$$

where K, G, H *are compatible linear triangulations of* C, D_1, N, $\# K$, $\# G$, $\# H \leqq d$; $H' \lhd H$; L *is a linear triangulation of* N, $\# H'$, $\# L \leqq D$ *and* f *is a simplicial isomorphism such that*

(*i*) $f_{|\partial N} = id$

(*ii*) $f(D_1) = D_2$

C *has the EPP if this is true at any polyhedral 2-cell* $D \subset \partial C$.

(*2*) *Any convex polyhedral 3-cell has the EPP*

This is true since proofs of Theorems 4–9 of Chap. 17 in [Mo] are (or can be easily reduced to) effective proofs.

(*3*) *Simply embedded spheres*

We say that a polyhedral sphere S is simply embedded if it bounds a polyhedral 3-cell B and ST_3 is true for B.

(*4*) *Theorem 17.11 in* [Mo]

Theorem. *Let S_1, S_2 be polyhedral 2-spheres in $\mathbb{R}^3$ such that $S_1 \cap S_2$ is a plane 2-cell D. Let*

$$S = (S_1 \cup S_2) \backslash \operatorname{int} D$$

If S_1 and S_2 are simply embedded, then so also is S.

Proof. See Theorem 17.11 in [Mo], using EPP instead of the push-property.

(*5*) *The index of a polyhedral 2-sphere*

Define the index of a polyhedral 2-sphere as in [Mo]: first choose a system $\mathscr{R}$ of orthogonal coordinates x, y, z in $\mathbb{R}^3$ in such a way that each plane $z = \text{const.}$ does not contain more than one vertex of S: this can be done effectively. For any horizontal plane E, $E \cap S$ can be: a single point, a (disconnected) 1-manifold, the disjoint union of a 1-manifold and a single point, the union of k polygons, 2 or more through a point P. In the latter case we call P a *singular* point and define the index at P, $\operatorname{Ind}_{\mathscr{R}} P$, to be $k - 1$. Then define $\operatorname{Ind}_{\mathscr{R}} S$, as the sum of the numbers $\operatorname{Ind}_{\mathscr{R}} P$ for all the singular points P in S.

(*6*) **Lemma.** *If S is a polyhedral sphere and $\operatorname{Ind}_{\mathscr{R}} S = n$, then one can effectively find a coordinate system $\mathscr{R}'$ and two polyhedral spheres S_1, S_2 such that S is simplicially isomorphic to their union (as in the step 4) and $\operatorname{Ind}_{\mathscr{R}}, S_i \leqq n - 1$ $i = 1, 2$.*

Proof. (see Lemma 17.1 of [Mo], p. 122). Let P be a singular point of the section $E \cap S$, where $E = \{z = z_E\}$ in the system $\mathscr{R}$, and let $J \subseteq E \cap S$ be the *inmost* polygon. J bounds a 2-cell D_J in E.

The only remark to be made is that the PL homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ constructed in the proof of [Mo], which preserves horizontal planes and makes D_J convex, is simplicial on an effectively computable triangulation of S and $\mathbb{R}^3$: but this is clear because of SS_2 and Lemma 2.5. □

Remark 2.25. SS_m and ST_m for $m = 4$, are open (to us), even under the restriction to 4-balls in $\mathbb{R}^4$.

The above facts suggest the following:

Problem 2.26 (embedding problem (E_m)). *It asks for*

$$\begin{cases} [K] \to [g\colon K' \to H] \\ [d] \to [D] \end{cases}$$

where $B = |K|$ is a triangulated PL m-ball, $\# K \leqq d$; $K' \lhd K$, $D \in \mathbb{N}$ and $\# K' \leqq D$, g is a simplicial isomorphism, $|H|$ is a PL m-ball embedded in $\mathbb{R}^m$.

3. On the effectiveness of the solutions of the real semialgebraic Hauptvermutung

We recall some well-known facts in semialgebraic geometry (see [BCR] or [BR]). We use the notations introduced in Sect. 0.

Theorem 3.1 (semialgebraic triangulation). *Let $X \in \mathscr{A}(n, s, D)$ be a compact semialgebraic set in $\mathbb{R}^n$. Let $Y_i \in \mathscr{A}(n, s, D)$ be a closed subset of X, $i = 1, \ldots, h$. Then*

$$\begin{cases} [X, Y_i] \to [K, f] \\ [n, s, D] \to [r, p, q, R] \end{cases}$$

where $|K|$ is a triangulated polyhedron of $\mathbb{R}^n$, $f: |K| \to X$ is a semialgebraic homeomorphism such that: $f^{-1}(Y_i) = |K_i|$ and K_i is a subcomplex of K, $i = 1, \ldots, h$; $\Gamma_f \in \mathscr{A}(p, q, R)$, $\# K \leqq r$.

We say that any $f: |K| \to X$ as above (without the limitation $|K| \subseteq \mathbb{R}^n$) is *a semialgebraic triangulation of X relative to $\{Y_i\}$.*

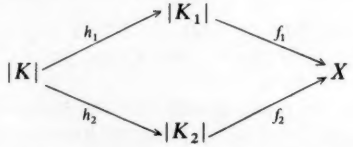
The following is an almost immediate corollary of 3.1.

Corollary 3.2. *Let $f_l: |K_l| \to X$ $l = 1, 2$ be a semialgebraic triangulation of X (relative to $\{Y_i\}$). Let $\# K_l \leqq d$, $\Gamma_{f_l} \in \mathscr{A}(n, s, D)$. Then*

$$\begin{cases} [f_1, f_2] \to [K, h_1, h_2] \\ [n, s, d, D] \to [r, m, q, R] \end{cases}$$

where for $l = 1, 2$ $h_l: |K| \to |K_l|$ is a semialgebraic triangulation of $|K_l|$ relative to the family of all simplexes of K_l, $\Gamma_{h_l} \in \mathscr{A}(m, q, R)$, $\# K \leqq r$.

Thus we have a diagram



and $(|K|, h_1, h_2)$ is called a simultaneous semialgebraic subdivision of the triangulations f_1, f_2 of X. Clearly $(|K|, f_l \circ h_l)$ $l = 1, 2$ is a semialgebraic triangulation of X.

(*A*) WEH_m

3.3. Proof of the WEH_m $\forall m$ (Theorem 2 of the Introduction). Let $f: |K| \to |L|$ be a semialgebraic homeomorphism between two polyhedra in $\mathbb{R}^n$ as in the statement of the theorem. Using Corollary 3.2 we may assume that f satisfies the further hypothesis: *for every simplex σ of K, $f(\sigma)$ is triangulated by a subcomplex of L.*

By Shiota-Yokoi existential solution of the Hauptvermutung we know that $B = f(\sigma)$ is a triangulated PL m-ball, $m = \dim \sigma$.

By means of Theorem 4 of the Introduction (see 2.17) we can effectively construct a trivialization of B. Thus we conclude (as in [SY], p. 737) by induction on $\dim |K|$, constructing effectively a PL isomorphism on the skeletons in increasing dimension and applying several times the "Alexander trick". □

Remark 3.4. The same proof as above (in the same way of [SY], p. 737) produces in fact a more precise result: that is *we can construct effectively a semialgebraic isotopy $H: |K| \times [0, 1] \to |L|$ such that $H_0 = f$ and $H_1 = g$, that is the simplicial isomorphism of the statement of* WEH_m.

Theorem 3.5 (Theorem 1 of the Introduction). *SEH_m is true for $m \leqq 3$.*

Proof. For $m \leqq 2$ we can work as in the Proof 3.3 using Proposition 2.23 instead of 2.17.

For $m = 3$ we want to reduce the Proposition 2.24, that is to work with 3-balls embedded in $\mathbb{R}^3$.

Consider a semialgebraic homeomorphism $f: |K| \to |L|$ such that for every simplex $\sigma \in K$, $B = f(\sigma)$ is triangulated by a subcomplex L_1 of L (like in the proof 3.3).

Take a 3-simplex σ of K, clearly we may consider $\sigma \subseteq \mathbb{R}^3$.

$g = f^{-1}: B = |L_1| \to \sigma$ is a semialgebraic triangulation of σ. Apply the semialgebraic triangulation theorem and its proof (see for instance [BCR] or [BR]) to σ relatively to the family given by the images of the simplexes of L_1. We obtain a straight triangulation of σ, say $\sigma = |H|$ such that every simplex of L_1 goes by g' over the support of a subcomplex of H, where $g': B \to \sigma$ is a suitable semialgebraic homeomorphism of effectively bounded complexity.

Thus we can effectively construct a PL isomorphism between $|L_1|$ and σ working like in the Proof 3.3 (induction, construction on skeletons of increasing dimension, Alexander trick) applying 2.25 to each $g'(\tau) \subseteq \sigma$, τ varying among the 3-simplexes of L_1. □

Remark 3.6. (in analogy with Remark 3.4.). For $m \leqq 3$ we can construct the semialgebraic isotopy H_t in such a way that $\Gamma_H \in \mathscr{A}(k, p, q)$ where k, p, q, are effectively bounded by the data of the situation.

(*B*) *SEH_m and semialgebraic homeomorphism of bounded complexity.*

At the present we are not able to decide if SEH_m holds for $m \geqq 4$.

In this section we show simply that SEH_m and an other natural semialgebraic problem cannot be solved effectively at the same time. The spirit of the result is analogous to Nabutovsky's examples.

Lemma 3.7. *For every $m, n, d \in \mathbb{N}$ there exist $P, Q \in \mathbb{N}$ such that P, Q depend on n, m, d and for every X, $Y \in \mathscr{A}(m, n, d)$ which are semialgebraically homeomorphic, there exists a semialgebraic homeomorphism $h: X \to Y$ with $\Gamma_h \in \mathscr{A}(m^2, P, Q)$.*

Proof. It is essentially the same of 2.22, using the local triviality theorem of semialgebraic maps. □

Problem 3.8. *Can P, Q as above be bounded by a recursive function of n, m, d?*

Proposition 3.9. *For $m \geqq 5$, the statement: "SEH_m has solution $\Rightarrow$ Problem 3.8 has negative answer" is true.*

Proof. Let $M = |K|$ be any compact polyhedron. If 3.8 has a positive solution we can effectively determine P, Q, depending only on m and $\# K$, such that if M is homeomorphic with Δ_m, then there exists a semialgebraic homeomorphism $h: M \to \Delta_m$ with $\Gamma_h \in \mathscr{A}(m^2, P, Q)$. If also SEH_m has solution we can deduce that one can effectively determine D depending only on $\# K$ and m such that if M is PL isomorphic with Δ_m then there exists a simplicial isomorphism $g: K' \to L$ where $K' \leqq K$, $L \lhd K_\Delta$ and $\# K' \leqq D$. This contradicts 2.18. □

Remark 3.10. The hard part in Shiota-Yokoi solution of subanalytic Hauptvermutung is in showing that if a polyhedron $P = |K|$ is subanalytically homeomorphic to Δ_m, then it is PL isomorphic to Δ_m (see also the Proof of 3.3 of the present paper). Shiota-Yokoi proof doesn't get any essential simplification in the semialgebraic

setting and it is far from giving SEH_m, not even WEH_m. The main reason of it is that Shiota-Yokoi use Cairns-Whithehead C^∞-triangulations and the (already known) solution of the Hauptvermutung for this class of triangulations (see [Mu]). This produce at least two troubles of increasing difficulty:

(*a*) It is not clear if a semialgebraic manifold $M \in \mathscr{A}(n, r, p)$ could be triangulated in the C^∞-sense, $M = |K|$ say, in such a way that $\# K$ is effectively controlled by n, r, q. The usual way (secant approximation . . . (see [Mu])) doesn't work: consider for example a sequence of manifolds $M_S \in \mathscr{A}(n, r, q)$ converging to some cusps as in the figure. The semialgebraic triangulation algorithm (again for the cusp presence) doesn't produce C^∞-triangulations.

(*b*) (More serious) Shiota-Yokoi apply the uniqueness of C^∞-triangulations to the diffeomorphisms obtained by *integrating vector fields*. Even if these vector fields are defined by semialgebraic functions of bounded complexity, how to bound, or even how to define, the *complexity* of the resulting diffeomorphism is a hard problem.

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A new characterization of rational surface singularities*

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I. Introduction

Let (R, m) be an excellent normal local domain of dimension two. Suppose that R/m is algebraically closed. In this paper it is shown that R has a rational singularity if and only if I^2 is a complete ideal for any m-primary complete ideal I of R. We give an example to show that this theorem is false if R/m is not algebraically closed.

Related problems have been considered in [Z1; ZS; MS; L; G; HS]. This paper is a continuation of my study of complete ideals in [C; C1] and [C2].

Section III gives an example of a normal but not Cohen-Macaulay scheme which birationally dominates a regular local ring of dimension three.

My main results are the following two theorems.

Theorem 1. *Suppose that (R, m) is an excellent and normal*[1] *local domain of dimension 2. Suppose that $k = R/m$ is algebraically closed. Then the following are equivalent.*

(1) R has a rational singularity.
(2) If I and J are any two complete ideals in R, then the product IJ is a complete ideal.
(3) If I and J are any two complete m-primary ideals in R, then the product IJ is a complete ideal.
(4) If I is any m-primary complete ideal in R, then I^2 is a complete ideal.

(1) implies (2) is proved by Lipman in [L].

However, this theorem fails to be true if k is not algebraically closed, as is shown in Theorem 2.

Theorem 2. *Let k be a field of characteristic not equal to 3. Let $R(k)$ be the ring*

$$R(k) = k[[x, y, z]]/x^3 + 3y^3 + 9z^3 .$$

* Partially supported by NSF

[1] The referee has pointed out that analytically normal is enough. The proof given here also works in this case, since we can replace R by its completion

The following is true

(*1*) *$R(k)$ is a normal local domain of dimension two and a simple elliptic singularity. In particular, it is not rational.*

(*2*) *Let $k = \mathbf{Q}$ be the rational numbers. If $I, J \subset R(\mathbf{Q})$ are any two complete ideals, then IJ is a complete ideal.*

(*3*) *If k has positive characteristic, or if k is an algebraically closed field, then there exists an m-primary complete ideal $I \subset R(k)$ such that I^2 is not a complete ideal.*

Theorem 1 is proved in Sect. I and Theorem 2 is proved in Sect. II.

Section III is devoted to giving an example showing interesting behavior in dimension three.

Example. There exists a normal but not Cohen-Macaulay scheme Z obtained by blowing up an m-primary ideal I in $R = \mathbf{C}[[x, y, z]]$ with natural map $h: Z \to \operatorname{spec}(R)$ such that $R^1h_*\mathcal{O}_Z = 0$ and $R^2h_*\mathcal{O}_Z \neq 0$. $\bigoplus_{n \geq 0} I^n$ is normal but not S_3.

For clarification, we will give an indication of the main idea of the proof of 4) implies 1) in Theorem 1.

Suppose that $f: X \to \operatorname{spec}(R)$ is a resolution of singularities, $I \subset R$ is an m-primary complete ideal, and $I\mathcal{O}_X = \mathcal{O}_X(D)$ is a Cartier divisor. Then I^2 is a complete ideal if and only if the natural map $H^0(\mathcal{O}_X(D))^{\otimes 2} \to H^0(\mathcal{O}_X(2D))$ is a surjection.

Suppose that R is such that it has the particularly simple form that there exists a resolution of singularities $f: X \to \operatorname{spec}(R)$ such that the reduced exceptional locus of f is irreducible. Call the reduced exceptional locus C. It is well known that R has a rational singularity if and only if C is a nonsingular rational curve.

Assume that R satisfies the assumptions of *4*) of Theorem 1, but that R does not have a rational singularity. We will derive a contradiction.

By Lemma 5 below, there exists a nonsingular closed point P on C and $\beta > 0$ such that if $\mathcal{L} = \mathcal{O}_C(\beta P)$, then $\mathcal{L}$ is generated by global sections, $H^1(\mathcal{L}) = H^1(\mathcal{L}^{\otimes 2}) = 0$, and $H^0(\mathcal{L})^{\otimes 2} \to H^0(\mathcal{L}^{\otimes 2})$ is not surjective. There exist $r > 0$ and nonsingular points $Q_1, \ldots, Q_s$ of C such that $\mathcal{O}_X(-rC) \otimes \mathcal{O}_C(-\beta P) \simeq \mathcal{O}_C(Q_1 + \ldots + Q_s)$. For simplicity, we will suppose that the Q_i are all distinct.

Let $g: Y \to X$ be the blow up of $Q_1, \ldots, Q_s$ with exceptional divisors $F_1, \ldots, F_s$. Set $D = g^*(-rC) - F_1 - \ldots - F_s$. Let $C' \simeq C$ be the strict transform of C by g. By construction, $\mathcal{O}_Y(D) \otimes \mathcal{O}_{C'} \simeq \mathcal{L}$. $\mathcal{O}_Y(D)$ is generated by global sections, and $H^0(\mathcal{O}_Y(D)) \to H^0(\mathcal{O}_{C'}(D))$, $H^0(\mathcal{O}_Y(2D)) \to H^0(\mathcal{O}_{C'}(2D))$ are surjections. $H^0(\mathcal{O}_Y(D))^{\otimes 2} \to H^0(\mathcal{O}_Y(2D))$ cannot be a surjection since $H^0(\mathcal{L})^{\otimes 2} \to H^0(\mathcal{L}^{\otimes 2})$ is not a surjection. $I = H^0(\mathcal{O}_Y(D))$ is an m-primary complete ideal since D has exceptional support. Hence $I^2 \neq \overline{I^2} = H^0(\mathcal{O}_Y(2D))$, a contradiction.

I thank Bill Heinzer for kindling my interest in the problem addressed in this paper.

Notations and some definitions are given. Then we will prove Theorem 1.

Notations are as in [H; A; L; C1; C2]. To every Weil divisor D on a normal scheme X there is an associated reflexive rank one sheaf which we will denote by $\mathcal{O}_X(D)$. The theory of sheaves of Weil divisors is developed in [*R*].

Definition 1. Suppose that $f: X \to Y$ is a proper morphism where X is a normal scheme. A Cartier Divisor D on X is called numerically effective for f if $(D \cdot C) \geqq 0$ for every curve C contracted by f.

Definition 2. (in [ZS] Appendix 4) Let R be a normal domain and let $I \subset R$ be an ideal. An element $z \in R$ is said to be integrally dependent on I if it satisfies an equation of the form

$$z^q + a_1 z^{q-1} + a_2 z^{q-2} + \ldots + a_q = 0 \quad \text{with} \quad a_i \in I^i .$$

The completion of I, denoted by $\bar{I}$, is the set of all $z \in R$ which are integrally dependent on I. I is called a complete ideal if $I = \bar{I}$.

We will make use of basic properties of complete ideals in [ZS] and [L].

Definition 3. Let (R, m) be a complete normal local domain of dimension two. We will say that R satisfies condition $(*)$ if the product I^2 is a complete ideal for any complete m-primary ideal I in R.

Lemma 1. *Suppose that R satisfies $(*)$. Let $f: X \to \operatorname{spec}(R)$ be a birational projective map such that X is normal. Suppose that D is an effective Cartier divisor with exceptional support for f, and that $\mathcal{O}_X(-D)$ is generated by global sections. Then*

$$H^0(\mathcal{O}_X(-D)) \otimes H^0(\mathcal{O}_X(-D)) \to H^0(\mathcal{O}_X(-2D))$$

is a surjection.

Proof. $I = H^0(\mathcal{O}_X(-D))$ is a complete m-primary ideal, and the completion of I^2 is $H^0(\mathcal{O}_X(-2D))$ by section five of [L].

Lemma 2. *Suppose that (R, m) is an excellent normal local domain, and that $I \subset R$ is an ideal. Let $\hat{R}$ be the completion of R with respect to m. Then I is a complete ideal in R if and only if $\hat{I}$ is a complete ideal in $\hat{R}$.*

Proof. Let $f: X \to \operatorname{spec}(R)$ be a birational projective map such that X is normal, and $I\mathcal{O}_X$ is invertible. Let

$$f': X' = X \otimes \hat{R} \to \operatorname{spec}(\hat{R})$$

be the natural map. X' and $\hat{R}$ are normal. I is complete if and only if $I = f_*(I\mathcal{O}_X)$ and $\hat{I}$ is complete if and only if $f'_*(\hat{I}\mathcal{O}_{X'}) = \hat{I}$. But $f'_*(\hat{I}\mathcal{O}_{X'}) = f_*(I\mathcal{O}_X) \otimes \hat{R}$ and $\hat{I} = I \otimes \hat{R}$, hence the lemma follows since completion is faithfully flat.

Lemma 3. *Suppose that (R, m) is a normal local domain of dimension two and that $k = R/m$ is algebraically closed. Suppose that $X \to \operatorname{spec}(R)$ is a projective birational morphism such that X is normal. Let EX be the reduced exceptional locus. Suppose that $C \subset X$ is an irreducible curve with exceptional support. Then there exists a positive integer $b(C)$ such that if $\mathcal{L}$ is a line bundle on C with $\deg(\mathcal{L}) > b(C)$, then $\mathcal{L}$ is generated by global sections, $H^1(\mathcal{L}) = 0$, and $\mathcal{L} \simeq \mathcal{O}_C(Q_1 + \ldots + Q_t)$ for some $t > 0$ where the Q_i are nonsingular points of both X and EX. If $\mathcal{F}$ is a coherent sheaf on C then there exists a positive integer $b(\mathcal{F})$ such that if $\mathcal{L}$ is a line bundle on C with $\deg(\mathcal{L}) > b(\mathcal{F})$, then $H^1(\mathcal{F} \otimes \mathcal{L}) = 0$.*

Proof. Given a coherent sheaf $\mathcal{F}$ on C we will first find an integer $b(\mathcal{F})$ such that $H^1(\mathcal{F} \otimes \mathcal{L}) = 0$ if $\mathcal{L}$ is a line bundle on C with $\deg(\mathcal{L}) > b(\mathcal{F})$. Let C' be the normalization of C. There is an exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C'} \to \mathcal{H} \to 0$$

where $\mathcal{H}$ has finite support. There is an exact sequence

$$0 \to \underline{\operatorname{Hom}}_{\mathcal{O}_C}(\mathcal{H}, \mathcal{F}) \to \underline{\operatorname{Hom}}_{\mathcal{O}_C}(\mathcal{O}_{C'}, \mathcal{F}) \xrightarrow{v} \underline{\operatorname{Hom}}_{\mathcal{O}_C}(\mathcal{O}_C, \mathcal{F}) \simeq \mathcal{F} \to \operatorname{coker}(v) \to 0$$

Set $\mathscr{D} = \mathrm{Hom}_{\mathscr{O}_C}(\mathscr{O}_{C'}, \mathscr{F})$. $\mathscr{D}$ is an $\mathscr{O}_C$, module. coker(v) has finite support. Hence $H^1(\mathrm{coker}(v) \otimes \mathscr{L}) = 0$ for any line bundle $\mathscr{L}$. There are exact sequences

$$0 \to \ker(v) \to \mathscr{D} \to \mathrm{image}(v) \to 0$$
$$0 \to \mathrm{image}(v) \to \mathscr{F} \to \mathrm{coker}(v) \to 0 \tag{1}$$

There exists $a > 0$ such that if $\mathscr{N}$ is a line bundle on C' with $\deg_{C'}(\mathscr{N}) > a$, then $H^1(C', \mathscr{D} \otimes \mathscr{N}) = 0$ by II.5.18 and III.7.12.1 [H]. Set $b(\mathscr{F}) = a$. (1) shows that if $\mathscr{L}$ is a line bundle on C with $\deg(\mathscr{L}) > b(\mathscr{F})$, then $H^1(\mathscr{F} \otimes \mathscr{L}) = 0$.

Now we will demonstrate the existence of $b(C)$. Let $P_1, \ldots, P_r$ be the singular points of EX and X contained in C. Let $m_{P_1}, \ldots, m_{P_r}$ be their ideals sheaves. Let

$$b(C) = \max\{b(\mathscr{O}_C) + 1, b(m_{P_1}), \ldots, b(m_{P_r}), b(m_{P_1} m_{P_2} \ldots m_{P_r})\}.$$

If $\mathscr{L}$ is a line bundle on C with $\deg(\mathscr{L}) > b(C)$, then $H^1(\mathscr{L}) = 0$ and $H^1(\mathscr{L} \otimes m_Q) = 0$ for any point $Q \in C$ with ideal sheaf m_Q. Hence $\mathscr{L}$ is generated by global sections.

$$H^0(\mathscr{L}) \to \mathscr{L}/(m_{P_1} m_{P_2} \ldots m_{P_r}) \otimes \mathscr{L}$$

is a surjection, which implies that there exists a global section σ of $\mathscr{L}$ whose support is disjoint from $P_1 + \ldots + P_r$. Hence the divisor of σ is $Q_1 + \ldots + Q_t$ where Q_i are nonsingular points of both X and EX, and $\mathscr{L} \simeq \mathscr{O}_C(Q_1 + \ldots + Q_t)$.

We will introduce some notation for use in Definitions 4 and 5 and Lemma 4. Suppose that (R, m) is a normal local domain of dimension two such that $k = R/m$ is algebraically closed and that $f: X \to \mathrm{spec}(R)$ is birational and projective where X has rational singularities. Let EX be the reduced exceptional locus of f.

Suppose that $EX = E_1 \cup \ldots \cup E_t \cup C_1 \cup \ldots \cup C_s$ where the E_i and C_j are the distinct irreducible components of EX, $\mathrm{sing}(X) \subset \cup E_i$, and that $\mathrm{sing}(C_j) \subset \cup E_i$ for all j. Suppose that H is an effective Cartier divisor on X with exceptional support and that $-H$ is numerically effective on X for f.

Definition 4. A decomposition sequence for H is defined to be a datum:

$$\mathscr{C} = (D_0, \ldots, D_r)$$

such that

$$D_0 = E$$
$$D_1 = E + C_{\sigma(1)}$$
$$\vdots$$
$$D_r = D_{r-1} + C_{\sigma(r)} = H$$

Where the support of E is $\cup E_i$ and $\sigma: [1, 2, \ldots, r] \to [1, \ldots, s]$.

Definition 5. Suppose that $\mathscr{C}$ is a decomposition sequence for H. Then we will define associated sheaves $\mathscr{F}_i$ for $\mathscr{C}$ and associated constants for $\mathscr{C}$

$$v = v(\mathscr{C}), \quad b_1 = b_1(\mathscr{C}), \ldots, b_r = b_r(\mathscr{C})$$

by the following construction.

Let $H = \Sigma \alpha_i C_i + \Sigma \beta_j E_j$. Given $p \in EX$ such that p is not contained in any E_i, define

$$v(p) = \Sigma\, \alpha_i \delta_i$$

where

$$\delta_i = \begin{cases} 1 \text{ if } p \in C_i \\ 0 \text{ if } p \text{ is not in } C_i \end{cases}$$

Let $v = v(\mathscr{C}) = \max_{p \in EX - \cup E_i}(v(p))$. Define coherent $\mathcal{O}_{C_{\sigma(i)}}$ modules $\mathscr{F}_i$ by

$$\mathscr{F}_i = \mathcal{O}_X(-D_{i-1})/\mathcal{O}_X(-D_i) = \mathcal{O}_X(-D_{i-1})/\mathcal{O}_X(-D_{i-1} - C_{\sigma(i)})$$

for $1 \leqq i \leqq r$. There exists positive integers $b_i = b(\mathscr{C})$ such that if $\mathscr{N}$ is a line bundle on $C_{\sigma(i)}$ with $\deg_{C_{\sigma(i)}}(\mathscr{N}) > b_i$, then $H^1(\mathscr{F}_i \otimes \mathscr{N}) = 0$ by Lemma 3. Note that $\mathcal{O}_X(-D_{i-1} - mH)/\mathcal{O}_X(-D_i - mH) \simeq \mathscr{F}_i \otimes \mathcal{O}_X(-mH)$.

Lemma 4. *Suppose that R satisfies condition $(*)$, $k = R/m$ is algebraically closed, and that $f: X \to \operatorname{spec}(R)$ is a birational projective morphism such that X has rational singularities. Suppose that H is a Cartier divisor on X with exceptional support such that $-H$ is numerically effective. Suppose that H has a decomposition sequence $\mathscr{C} = (D_0 = E, \ldots, H = D_r)$ with associated constants $v = v(\mathscr{C})$ and $b_i = b_i(\mathscr{C})$ as in Definitions 4 and 5. Suppose that $\mathscr{L}$ is a line bundle on X such that*

(1) $\deg_{C_j}(\mathscr{L}) > (\max_{\sigma(i)=j} b_i) + v$ *for all* j.
(2) $H^1(\mathcal{O}_E(-mH) \otimes \mathscr{L}^{\otimes k}) = 0$ *for all* $m \geqq 0$ *and* $k = 1$ *or* 2.
(3) $\mathscr{L} \otimes \mathcal{O}_E$ *is generated by global sections.*
(4) $H^0(\mathscr{L} \otimes \mathcal{O}_E) \otimes H^0(\mathscr{L} \otimes \mathcal{O}_E) \to H^0(\mathscr{L}^{\otimes 2} \otimes \mathcal{O}_E)$ *is not surjective.*

Then

(A) $\mathscr{L}$ *is generated by global sections*
(B) $H^0(\mathscr{L}) \otimes H^0(\mathscr{L}) \to H^0(\mathscr{L}^{\otimes 2})$ *is not surjective.*

Proof. Consider for $m \geqq 0$ and $\mathscr{N} = \mathscr{L}$ or $\mathscr{L}^{\otimes 2}$ the following exact sequences.

$$0 \to \mathscr{F}_1 \otimes \mathcal{O}_X(-mH) \otimes \mathscr{N} \to \mathcal{O}_{mH+D_1} \otimes \mathscr{N} \to \mathcal{O}_{mH+E} \otimes \mathscr{N} \to 0$$

$$\vdots$$

$$0 \to \mathscr{F}_i \otimes \mathcal{O}_X(-mH) \otimes \mathscr{N} \to \mathcal{O}_{mH+D_i} \otimes \mathscr{N} \to \mathcal{O}_{mH+D_{i-1}} \otimes \mathscr{N} \to 0$$

$$\vdots$$

$$0 \to \mathscr{F}_r \otimes \mathcal{O}_X(-mH) \otimes \mathscr{N} \to \mathcal{O}_{(m+1)H} \otimes \mathscr{N} \to \mathcal{O}_{mH+D_{r-1}} \otimes \mathscr{N} \to 0$$

$$0 \to \mathcal{O}_E(-mH) \otimes \mathscr{N} \to \mathcal{O}_{mH+E} \otimes \mathscr{N} \to \mathcal{O}_{mH} \otimes \mathscr{N} \to 0$$

Assumption *(1)* of the lemma implies that $H^1(\mathscr{F}_i \otimes \mathcal{O}_X(-mH) \otimes \mathscr{N}) = 0$ for $m \geqq 0$ and $1 \leqq i \leqq r$. Assumption *(2)* implies that $H^1(\mathcal{O}_E(-mH) \otimes \mathscr{N}) = 0$ for $m \geqq 0$. Hence $H^0(\mathcal{O}_{mH} \otimes \mathscr{N}) \to H^0(\mathcal{O}_E \otimes \mathscr{N})$ is a surjection for all $m > 0$, and $H^0(\mathscr{N}) = \varprojlim H^0(\mathcal{O}_{mH} \otimes \mathscr{N}) \to H^0(\mathcal{O}_E \otimes \mathscr{N})$ is a surjection. In particular, by assumption *4)*, $H^0(\mathscr{L}) \otimes H^0(\mathscr{L}) \to H^0(\mathscr{L}^{\otimes 2})$ is not a surjection, and by assumption *3)*, $\mathscr{L}$ is generated by global sections at points contained in E.

Suppose that $p \in EX$ is a closed point such that p is not contained in any E_i. By assumption, p is a nonsingular point of X. Let $g: Y \to X$ be the blow up of p with exceptional curve L. Let C_i' be the proper transform of C_i by g, and let $H' = g^*(H)$. Identify $g^*(E)$ with E. Set $a = v(p) \leqq v$. Recall that $v(p)$ is defined in Definition 5.

H' has a decomposition sequence

$$\begin{aligned} D'_0 &= aL \\ D'_1 &= aL + E \\ D'_2 &= D'_1 + C'_{\sigma(1)} \\ &\vdots \\ D'_i &= D'_{i-1} + C'_{\sigma(i-1)} \\ D'_{i+1} &= D'_i + C'_{\sigma(i)} \\ &\vdots \\ H' &= D'_{r+1} = D'_r + C'_{\sigma(r)} \end{aligned}$$

We have

$$\begin{aligned} D'_1 &= g^*(E) + aL \\ &\vdots \\ D'_i &= g^*(D_{i-1}) + (b_{i-1})L \\ D'_{i+1} &= g^*(D_i) + b_i L \quad \text{for some} \quad b_i \quad \text{with} \quad 0 \leqq b_i \leqq a \\ &\vdots \\ H' &= D'_{r+1} = g^*(D_r) \end{aligned}$$

Set $\mathcal{N} = g^*(\mathcal{L})$. Clearly $\mathcal{O}_{aL}(-mH') \otimes \mathcal{N} \simeq \mathcal{O}_{aL}$ $H^1(\mathcal{O}_{aL}(-mH') \otimes \mathcal{N}) = 0$ for all m, since $L \simeq P^1$ and $(L)^2 = -1$. This shows that $H^0(\mathcal{O}_{aL} \otimes \mathcal{N}) \to H^0(\mathcal{O}_L \otimes \mathcal{N}) = H^0(\mathcal{O}_L)$ is a surjection.

For $1 \leqq i \leqq r$ define coherent $\mathcal{O}_{C'_{\sigma(i)}}$ modules $\mathcal{D}_i$ by

$$\mathcal{D}_i = \mathcal{O}_Y(-D'_i)/\mathcal{O}_Y(-D'_i - C'_{\sigma(i)}) .$$

Consider for $m \geqq 0$ the following exact sequences

$$0 \to \mathcal{O}_{aL}(-mH') \otimes \mathcal{N} \to \mathcal{O}_{mH'+aL} \otimes \mathcal{N} \to \mathcal{O}_{mH'} \otimes \mathcal{N} \to 0 \tag{2}$$

$$0 \to \mathcal{O}_E(-mH' - aL) \otimes \mathcal{N} \simeq \mathcal{O}_E(-mH) \otimes \mathcal{L} \to \mathcal{O}_{mH'+E+aL} \otimes \mathcal{N} \to \mathcal{O}_{mH'+aL} \otimes \mathcal{N} \to 0$$

$$0 \to \mathcal{D}_1 \otimes \mathcal{O}(-mH') \otimes \mathcal{N} \to \mathcal{O}_{mH'+D'_2} \otimes \mathcal{N} \to \mathcal{O}_{mH'+D'_1} \otimes \mathcal{N} \to 0$$

$$\vdots$$

$$0 \to \mathcal{D}_i \otimes \mathcal{O}_Y(-mH') \otimes \mathcal{N} \to \mathcal{O}_{mH'+D'_{i+1}} \otimes \mathcal{N} \to \mathcal{O}_{mH'+D'_i} \otimes \mathcal{N} \to 0$$

$$\vdots$$

$$0 \to \mathcal{D}_r \otimes \mathcal{O}_Y(-mH') \otimes \mathcal{N} \to \mathcal{O}_{(m+1)H'} \otimes \mathcal{N} \to \mathcal{O}_{mH'+D'_r} \otimes \mathcal{N} \to 0$$

If p is not contained in the support of $C'_{\sigma(i)}$ then $\mathcal{D}_i \simeq \mathcal{F}_i$. If p is contained in the support of $C'_{\sigma(i)}$ then $\mathcal{D}_i \simeq \mathcal{F}_i \otimes \mathcal{O}_{C_{\sigma(i)}}(-[b_{i-1}]p)$. Hence we have inclusions $\mathcal{F}_i \otimes \mathcal{O}_X(-mH) \otimes \mathcal{O}_{C_{\sigma(i)}}(-vp) \otimes \mathcal{L} \to \mathcal{D}_i \otimes \mathcal{O}_Y(-mH') \otimes \mathcal{N}$ for $1 \leqq i \leqq r$, where $\mathcal{O}_{C_j}(-vp)$ is defined to be $\mathcal{O}_{C_j}$ if p is not contained in the support of C_j. From assumption (*1*) it follows that $H^1(\mathcal{D}_i \otimes \mathcal{O}_Y(-mH') \otimes \mathcal{N}) = 0$ for $1 \leqq i \leqq r$ and $m \geqq 0$. Assumption (*2*) shows that $H^1(\mathcal{O}_E(-mH') \otimes \mathcal{L}) = 0$ for all $m \geqq 0$. Consideration of the exact sequences (2) now shows that $H^0(\mathcal{N}) = H^0(\mathcal{L}) \to$

$H^0(\mathcal{O}_L \otimes \mathcal{N}) = \mathcal{L}/m_p\mathcal{L}$ is a surjection, where m_p is the maximal ideal of $\mathcal{O}_{X,p}$, and $\mathcal{L}$ is generated by global sections at p. Since $\mathcal{L}$ is generated by global sections at all points contained in EX, $\mathcal{L}$ is generated by global sections.

Lemma 5. *Suppose that C is an irreducible reduced (but maybe not normal) curve over an algebraically closed field k with $h^1(\mathcal{O}_C) > 0$. Then there exists a line bundle $\mathcal{L}$ on C such that*

(1) $\mathcal{L}$ is generated by global sections
(2) $H^1(\mathcal{L}) = H^1(\mathcal{L}^{\otimes 2}) = 0$
(3) $H^0(\mathcal{L}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2})$ is not surjective.

Proof. We will use the notation of II §7 [H]. If $Q \in C$ is a closed point, let m_Q be its ideal sheaf. Let $P \in C$ be a closed nonsingular point. There are only a finite number of singular points $Q_1, \ldots, Q_s$ on C. If $Q \in C$ is not a singular point then m_Q is a line bundle of degree -1. Hence by Lemma 3 there exists a smallest $\beta \geqq 1$ such that $H^1(\mathcal{O}_C(\beta P) \otimes m_Q) = 0$ for all $Q \in C$. Let $\mathcal{L} = \mathcal{O}_C(\beta P)$. Clearly $\beta > 1$ since $H^1(\mathcal{O}_C(P) \otimes m_P) = H^1(\mathcal{O}_C) \neq 0$. First we will show that $|\beta P|$ is base point free. Given $Q \in C$, there is an exact sequence

$$0 \to m_Q \otimes \mathcal{O}_C(\beta P) \to \mathcal{O}_C(\beta P) \to \mathcal{O}_C(\beta P)/m_Q\mathcal{O}_C(\beta P) \to 0$$

Hence $H^0(\mathcal{O}_C(\beta P)) \to \mathcal{O}_C(\beta P)/m_Q\mathcal{O}_C(\beta P)$ is a surjection, and $\mathcal{O}_C(\beta P)$ is generated by global sections.

Now we will show that there exists $Q \in C$ such that $|\beta P|$ does not separate P and Q. We know that $H^1(\mathcal{O}_C(\beta P) \otimes m_P) = H^1(\mathcal{O}_C((\beta - 1)P)) = 0$, but there exists a closed point $Q \in C$ such that $H^1(\mathcal{O}_C((\beta - 1)P) \otimes m_Q) \neq 0$. Consideration of

$$0 \to m_Q \otimes \mathcal{O}_C((\beta - 1)P) \to \mathcal{O}_C((\beta - 1)P) \to \mathcal{O}_C((\beta - 1)P)/m_Q\mathcal{O}_C((\beta - 1)P) \to 0 \quad (3)$$

shows that $H^0(\mathcal{O}_C((\beta - 1)P)) \to \mathcal{O}_C((\beta - 1)P)/m_Q\mathcal{O}_C((\beta - 1)P)$ is not a surjection. There is an inclusion

$$\mathcal{O}_C(\beta P) \otimes m_Q \to \mathcal{O}_C((2\beta - 1)P \otimes m_Q)$$

$H^1(\mathcal{O}_C(\beta P) \otimes m_Q) = 0$ implies that $H^1(\mathcal{O}_C((2\beta - 1)P) \otimes m_Q) = 0$. Tensoring (3) with $\mathcal{O}_C(\beta P)$ shows that $|2\beta P|$ separates P and Q. In particular $H^0(\mathcal{O}_C(\beta P)) \otimes H^0(\mathcal{O}_C(\beta P)) \to H^0(\mathcal{O}_C(2\beta P))$ cannot be surjective since every section of $H^0(\mathcal{O}_C(\beta P))$ containing P in its support must also contain Q.

Proposition 1. *Suppose that R satisfies condition $(*)$, $k = R/m$ is algebraically closed, and $f: X \to \operatorname{spec}(R)$ is birational, projective such that X has rational singularities. Let EX be the reduced exceptional locus of f. Suppose that $EX = C \cup C_1 \ldots \cup C_e$ where $C, C_1, \ldots, C_e$ are the distinct irreducible components of EX. Suppose that $\operatorname{sing}(X) \subset C$ and that $\operatorname{sing}(C_i) \subset C \cap C_i$ for $i = 1, \ldots, e$. Then $H^1(\mathcal{O}_C) = 0$.*

Proof. Suppose that $H^1(\mathcal{O}_C) \neq 0$. Let $b = b(C)$ be the number computed in Lemma 3. There exists a smallest integer $r_0 > 0$, such that $r_0 C$ is a Cartier divisor on X. Let $\alpha: Z \to X$ be a resolution of singularities. Let C'' be the proper transform of C by α. Then $\deg_C(\mathcal{O}_C(r_0 C)) = \deg_{C''}(\mathcal{O}_{C''}(g^*(r_0 C))) = (C'' \cdot (g^*(r_0 C)))$ $= \left(\frac{1}{r_0}\right)(g^*(r_0 C))^2 < 0$ since the intersection form for $Z \to \operatorname{spec}(R)$ is negative definite. Given $n \geqq 0$, write $n = mr_0 + t$ with $0 \leqq t < r_0$.

$$\mathcal{O}_X(-nC)/\mathcal{O}_X(-(n+1)C) \simeq [\mathcal{O}_X(-tC)/\mathcal{O}_X(-(t+1)C)] \otimes \mathcal{O}_C(-mr_0 C)$$

Set $d = \max\{b(C), b(\mathcal{O}_X(-C)/\mathcal{O}_X(-2C)), \ldots, b(\mathcal{O}_X(-(r_0 - 1)C)/\mathcal{O}_X(-r_0 C))\}$.

By Lemma 3, if $\mathscr{H}$ is a line bundle on C with $\deg(\mathscr{H}) > d$ then $H^1(\mathscr{H} \otimes [\mathscr{O}_X(-nC)/\mathscr{O}_X(-(n+1)C)]) = 0$ for all $n \geqq 0$.

Let $\gamma: X' \to X$ be the blow up of general closed points $W_1, \ldots, W_{d+1}$ of C. Let $C''' \simeq C$ be the proper transform of C by γ.

$$\mathscr{O}_{X'}(-nC''')/\mathscr{O}_{X'}(-(n+1)C''') \simeq [\mathscr{O}_X(-nC)/\mathscr{O}_X(-(n+1)C)] \otimes \mathscr{O}_C(n(W_1 + \ldots + W_{d+1}))$$

Hence if $\mathscr{H}$ is a line bundle on C''' such that $\deg(\mathscr{H}) \geqq 0$ then $H^1(\mathscr{H} \otimes [\mathscr{O}_{X'}(-nC''')/\mathscr{O}_{X'}(-(n+1)C''')]) = 0$ for all $n > 0$. Without loss of generality we may replace X by X'. If $\mathscr{H}$ is a line bundle on C such that $\deg_C(\mathscr{H}) \geqq 0$, then

$$H^1([\mathscr{O}_X(-nC)/\mathscr{O}_X(-(n+1)C)] \otimes \mathscr{H}) = 0 \quad \text{for all} \quad n > 0\,. \tag{4}$$

Let $\mathscr{O}_C(\beta P)$ be the divisor constructed in the proof of Lemma 5. There exists $\lambda > 0$ such that

$$b - (C \cdot r_0 C) \geqq \lambda(C \cdot (-r_0 C)) - \beta > b$$

Set $r_1 = \lambda r_0$. Let $s = \lambda(C \cdot (-r_0 C)) - \beta$. There exists H', an effective ample Cartier divisor on X with exceptional support such that

$$(C \cdot (-H')) > b \tag{5}$$

and H' has a decomposition sequence $\mathscr{C}' = (D'_0, \ldots, D'_r)$ satisfying the assumptions of Definition 4, and which can be written as (6) below.

$$\begin{aligned} D'_0 &= aC \quad (\text{for some } a > 0) \\ D'_1 &= D'_0 + C_{\sigma(1)} \\ &\vdots \\ H' &= D'_r = D'_{r-1} + C_{\sigma(r)} \end{aligned} \tag{6}$$

Let $\mathscr{F}_i$ be the associated sheaves of $\mathscr{C}'$ and $b_i = b_i(\mathscr{C}')$ and $v = v(\mathscr{C}')$ be the associated constants for $\mathscr{C}'$ computed in Definition 5. There exists an effective Cartier divisor Δ on X with exceptional support such that $(\Delta \cdot C) = 0$, and $(C_j \cdot (-\Delta - r_1 C)) > (\max_{\sigma(i)=j} b_i) + v$.

By Lemma 3, there exist nonsingular points $Q_1, \ldots, Q_s$ of both X and EX such that $\mathscr{O}_C(-\Delta - \beta P - r_1 C) \simeq \mathscr{O}_C(Q_1 + \ldots + Q_s)$. Write $Q_1 + \ldots + Q_s = \sum_{i=1}^{t} d_i Q_i$ where $t \leqq s$, $\Sigma d_i = s$, and $Q_i \neq Q_j$ if $i \neq j$.

Let $gY \to X$ be the blow up of $Q_1, \ldots, Q_t$ with exceptional curves $F_1, \ldots, F_t$. Let C_i be identified with their proper transforms by g. Let C' be the proper transform of C by g. Let

$$D = g^*(-\Delta) - g^*(r_1 C) - d_1 F_1 - \ldots - d_t F_t$$

D is Cartier and has exceptional support for $Y \to \operatorname{spec}(R)$. By construction, $\mathscr{O}_Y(D) \otimes \mathscr{O}_{C'} \simeq \mathscr{O}_C(\beta P)$. $(D \cdot F_i) = d_i \geqq 1$, and

$$(D \cdot C_i) = ((-\Delta - r_1 C) \cdot C_i) > \left(\max_{\sigma(j)=i} b_j\right) + v\,. \tag{7}$$

for all i.

Let $H = g^*(H')$. H has a decomposition series $\mathscr{C} = (D_0, \ldots, D_r)$ as in Definition 4, obtained from (6) by applying g^*. $\mathscr{C}$ can be written as:

$$\begin{aligned} D_0 &= g^*(D_0') \\ D_1 &= g^*(D_1)' \\ &\vdots \\ H = D_r &= g^*(D_r') \end{aligned}$$

In the notation of Definition 5, $g^*(\mathscr{F}_1), \ldots, g^*(\mathscr{F}_r)$ are the associated sheaves of $\mathscr{C}$, and we can take $b_i = b_i(\mathscr{C}) = b_i(\mathscr{C}')$ for $1 \leqq i \leqq r$, $v = v(\mathscr{C}) = v(\mathscr{C}')$ to be the associated constants for $\mathscr{C}$, since g is an isomorphism in a neighborhood of $\cup C_i$.

Set $E = D_0 = a(C' + F_1 + \ldots F_t)$. Let $\mathscr{L} = \mathscr{O}_Y(D)$. We will show that the decomposition sequence $\mathscr{C}$ with its associated constants b_i and v, and $\mathscr{L}$ verify assumptions 1)–4) of Lemma 4. Assumption *1)* is verified in equation (7) above.

Consider for $m \geqq 0$, $a \geqq n \geqq 0$ and $\mathscr{N} = \mathscr{L}$ or $\mathscr{L}^{\otimes 2}$ the sheaves

$$\mathscr{F}(m, n) = \left\{ \left[\frac{\mathscr{O}_Y(-n(C' + F_1 + \ldots + F_t))}{\mathscr{O}_Y(-C' - n(C' + F_1 + \ldots + F_t))} \right] \otimes \mathscr{N} \otimes \mathscr{O}_Y(-mH) \right\}.$$

There are exact sequences:

$$\begin{aligned} 0 \to\ & \mathscr{O}_{F_1 + \ldots + F_t}(-C' - n(C' + F_1 + \ldots + F_t)) \otimes \mathscr{N} \otimes \mathscr{O}_Y(-mH) \to \\ & \mathscr{O}_{(n+1)(C' + F_1 + \ldots + F_t)} \otimes \mathscr{N} \otimes \mathscr{O}_Y(-mH) \to \\ & \mathscr{O}_{C' + n(C' + F_1 + \ldots + F_t)} \otimes \mathscr{N} \otimes \mathscr{O}_Y(-mH) \to 0 \qquad (8) \\ 0 \to\ & \mathscr{F}(m, n) \to \mathscr{O}_{C' + n(C' + F_1 + \ldots + F_t)} \otimes \mathscr{N} \otimes \mathscr{O}_Y(-mH) \to \\ \to\ & \mathscr{O}_{n(C' + F_1 + \ldots F_t)} \otimes \mathscr{N} \otimes \mathscr{O}_Y(-mH) \to 0 \\ & h^1(\mathscr{O}_{F_1 + \ldots + F_t}(-C' - n(C' + F_1 + \ldots + F_t)) \otimes \mathscr{N} \otimes \mathscr{O}_Y(-mH)) \\ = & \sum_{i=1}^{t} h^1(\mathscr{O}_{\mathbf{P}^1}(-1) \otimes \mathscr{O}_{\mathbf{P}^1}(cd_i)) = 0, \quad \text{where} \quad c = 1 \quad \text{or} \quad 2 \end{aligned}$$

depending on whether or not $\mathscr{N} = \mathscr{L}$ or $\mathscr{L}^{\otimes 2}$.

$$\begin{aligned} \mathscr{F}(m, n) &\simeq [\mathscr{O}_Y(-nC')/\mathscr{O}_Y(-(n+1)C')] \\ &\quad \otimes \mathscr{O}_Y(-n(F_1 + \ldots + F_t)) \otimes \mathscr{N} \otimes \mathscr{O}_Y(-mH) \\ &\simeq [\mathscr{O}_X(-nC)/\mathscr{O}_X(-(n+1)C)] \otimes \mathscr{N} \otimes \mathscr{O}_X(-mH') \end{aligned}$$

When $n > 0$, $H^1(\mathscr{F}(m, n)) = 0$ by (4). $\mathscr{F}(m, 0) = \mathscr{O}_C \otimes \mathscr{N} \otimes \mathscr{O}_X(-mH')$. Hence (5) shows that $H^1(\mathscr{F}(m, 0)) = 0$ for $m > 0$. $\mathscr{F}(0, 0) = \mathscr{O}_C \otimes \mathscr{N}$. $H^1(\mathscr{O}_C \otimes \mathscr{N}) = 0$ and $\mathscr{O}_C \otimes \mathscr{N}$ is generated by global sections by the choice of $\mathscr{L}$. Assumption *2)* of Lemma 4, and the fact that $\mathscr{L} \otimes \mathscr{O}_E$ is generated by global sections in a neighborhood of C' now follows from induction using the sequences (8). Assumption 4) of Lemma 4 is true by Lemma 5, since $H^0(\mathscr{O}_E \otimes \mathscr{N}) \to H^0(\mathscr{O}_{C'} \otimes \mathscr{N})$ is a surjection.

It remains to verify that $\mathscr{L} \otimes \mathscr{O}_E$ is generated by global sections in a neighborhood of each F_i. We need only check for F_1, since the argument is the same for each

F_i. For $m \geqq 0$, set

$$K_1(m) = [\mathcal{O}_Y(-F_1 - m(C' + F_1 + \ldots + F_t))/\mathcal{O}_Y(-F_1 - m(C' + F_1 + \ldots + F_t) - C')] \otimes \mathcal{L}$$

$$K_2 = \mathcal{O}_{F_2 + \ldots + F_t}(-C' - F_1 - m(C' + F_1 + \ldots + F_t)) \otimes \mathcal{L}$$

$$K_3 = \mathcal{O}_{F_1}(-(m+1)(C' + F_1 + \ldots + F_t)) \otimes \mathcal{L}$$

There are exact sequences:

$$0 \to K_1(m) \to \mathcal{O}_{C' + F_1 + m(C' + F_1 + \ldots + F_t)} \otimes \mathcal{L} \to \mathcal{O}_{F_1 + m(C' + F_1 + \ldots + F_t)} \otimes \mathcal{L} \to 0$$

$$0 \to K_2 \to \mathcal{O}_{(m+1)(C' + F_1 + \ldots + F_t)} \otimes \mathcal{L} \to \mathcal{O}_{C' + F_1 + m(C' + F_1 + \ldots + F_t)} \otimes \mathcal{L} \to 0$$

$$0 \to K_3 \to \mathcal{O}_{(m+1)(C' + F_1 + \ldots + F_t) + F_1} \otimes \mathcal{L} \to \mathcal{O}_{(m+1)(C' + F_1 + \ldots + F_t)} \otimes \mathcal{L} \to 0 \qquad (9)$$

$$K_1(m) \simeq [\mathcal{O}_Y(-mC')/\mathcal{O}_Y(-(m+1)C')] \otimes \mathcal{O}_Y(-m(F_1 + \ldots + F_t) - F_1) \otimes \mathcal{L}$$

$$\simeq [\mathcal{O}_X(-mC)/\mathcal{O}_X(-(m+1)C)] \otimes \mathcal{O}_C(-Q_1 + \beta P)$$

$H^1(K_1(m)) = 0$ for $m > 0$ by (4). $H^1(K_1(0)) = H^1(\mathcal{O}_C(\beta P - Q_1)) = 0$ by choice of $\mathcal{L}$. $K_2 \simeq \bigoplus_{i=2}^{t} \mathcal{O}_{\mathbf{P}^1}(d_i - 1)$. Hence $H^1(K_2) = 0$. $K_3 \simeq \mathcal{O}_{\mathbf{P}^1}(d_1)$ implies that $H^1(K_3) = 0$. In particular, consideration of the exact sequences (9) and induction shows that $H^0(\mathcal{O}_E \otimes \mathcal{L}) \to H^0(\mathcal{O}_{F_1} \otimes \mathcal{L}) \simeq H^0(\mathcal{O}_{\mathbf{P}^1}(d_1))$ is a surjection, and $\mathcal{O}_E \otimes \mathcal{L}$ is generated by global sections in a neighborhood of F_1.

Now Lemma 4 shows that $H^0(\mathcal{L}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2})$ is not a surjection, in contradiction to Lemma 1. This contradiction shows that $H^1(\mathcal{O}_C) = 0$.

Proposition 2. *Suppose that R satisfies condition (*), $k = R/m$ is algebraically closed, and that $f: X \to \operatorname{spec}(R)$ is a birational projective map where X has rational singularities. Suppose that EX is the reduced exceptional locus of f, and that C is an irreducible component of EX such that $\operatorname{sing}(X) \subset C$. Suppose that if $A \subset EX$ is an irreducible component, then $\operatorname{sing}(A) \subset C$. Then $C \simeq \mathbf{P}^1$, and $\mathcal{O}_X(-C)/\mathcal{O}_X(-2C) \simeq \mathcal{O}_{\mathbf{P}^1}(e)$ for some e with $e \geqq -1$.*

Proof. $C \simeq \mathbf{P}^1$ by Proposition 1, consideration of the normalization sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C'} \to \mathcal{F} \to 0$$

where C' is the normalization of C and $\mathcal{F}$ has finite support, and by IV.1.3.5 [H]. $\mathcal{O}_X(-C)/\mathcal{O}_X(-2C)$ is a line bundle on C since C is nonsingular and $\mathcal{O}_X(-C)/\mathcal{O}_X(-2C)$ is torsion free.

Suppose that $\mathcal{O}_X(-C)/\mathcal{O}_X(-2C) \simeq \mathcal{O}_{\mathbf{P}^1}(-s)$ with $s \geqq 2$. We will derive a contradiction. Let $\alpha: X' \to X$ be the blow up of $s+3$ general points on C with exceptional curves $F_1, \ldots, F_s, F_{s+1}, L_1, L_2$. Let F be the proper transform of C by α. Then $\mathcal{O}_{X'}(-F)/\mathcal{O}_{X'}(-2F) \simeq \mathcal{O}_{\mathbf{P}^1}(3)$.

Let $B' = 2F + F_1 + F_2 + F_3 + L_1 + L_2$. Let $\mathcal{L}$ be a numerically effective line bundle on B' such that $\deg_{L_i}(\mathcal{L}) \geqq 1$ for some i. We will show that $H^1(\mathcal{L}) = 0$. Without loss of generality, we may suppose that $\deg_{L_2}(\mathcal{L}) > 0$. There are exact sequences

$$0 \to [\mathcal{O}_{X'}(-F)/\mathcal{O}_{X'}(-2F)] \otimes \mathcal{O}_{X'}(-F_1 - F_2 - F_3 - L_1) \otimes \mathcal{L}$$

$$\simeq \mathcal{O}_{\mathbf{P}^1}(-1) \otimes \mathcal{L} \to \mathcal{O}_{2F + F_1 + F_2 + F_3 + L_1} \otimes \mathcal{L} \to$$

$$\mathcal{O}_{F + F_1 + F_2 + F_3 + L_1} \otimes \mathcal{L} \to 0$$

$$0 \to \mathcal{O}_{L_2}(-2F - F_1 - F_2 - F_3 - L_1) \otimes \mathcal{L} \simeq \mathcal{O}_{\mathbf{P}^1}(-2) \otimes \mathcal{L} \to$$
$$\mathcal{L} \to \mathcal{O}_{2F+F_1+F_2+F_3+L_1} \otimes \mathcal{L} \to 0 \tag{10}$$

It is straight forward to check that $H^1(\mathcal{O}_{F+F_1+F_2+F_3+L_1} \otimes \mathcal{L}) = 0$, hence $H^1(\mathcal{O}_{B'} \otimes \mathcal{L}) = 0$. (10) also shows that if $\deg_{L_i}(\mathcal{L}) \geqq 1$ for $i = 1$ and 2, then $\mathcal{L}$ is generated by global sections. A similar sequence shows that $H^1(\mathcal{O}_{B'}) \simeq k$.

Let α_1 and α_2 be general (closed) points of L_1 and L_2 respectively. There exists a Cartier divisor D' with exceptional support for $X' \to \operatorname{spec}(R)$ such that $(D' \cdot \gamma) = 0$ if γ is exceptional and $\gamma \neq L_1$ or L_2, but $(D' \cdot L_1) > 0$ and $(D' \cdot L_2) > 0$ since the intersection form for X' is negative definite.

Let $z_1 = (D' \cdot L_1) - 1$, $z_2 = (D' \cdot L_2) - 1$. Let $\beta: X'' \to X'$ be the blow up of α_1 and α_2 with exceptional divisors A_1 and A_2. Let $D'' = \beta^*(D') - z_1 A_1 - z_2 A_2$. D'' is numerically effective on X'' with exceptional support for $X'' \to \operatorname{spec}(R)$. Identify F, F_1, F_2, F_3, L_1 and L_2 with their proper transforms by β. $(D'' \cdot L_1) = (D'' \cdot L_2) = 1$, and $(D'' \cdot F) = (D'' \cdot F_1) = (D'' \cdot F_2) = (D'' \cdot F_3) = 0$.

Let $\gamma: Z \to X''$ be the blow up of general (closed) points on F_1, F_2, F_3, L_1, and L_2 so that if F, F_1, F_2, F_3, L_1 and L_2 are identified with their proper transforms by γ, then there exists $t > 0$ such that if $W = tF + F_1 + F_2 + F_3 + L_1 + L_2$, then W is Cartier, $(-W \cdot F) > 0$, $(-W \cdot F_1) > 0$, $(-W \cdot F_2) > 0$, $(-W \cdot F_3) > 0$, $(-W \cdot L_1) > 0$, and $(-W \cdot L_2) > 0$. Let $B = 2F + F_1 + F_2 + F_3 + L_1 + L_2$. Let $D = \gamma^*(D'')$. Let H be an effective Cartier divisor on Z with exceptional support for $Z \to \operatorname{spec}(R)$ such that $-H$ is ample. H has a decomposition sequence $\mathscr{C} = (D_0, \ldots, D_r)$ as in Definition 5, such that $\mathscr{C}$ can be written as:

$$D_0 = E$$
$$D_1 = E + C_{\sigma(1)}$$
$$\vdots$$
$$H = D_r = D_{r-1} + C_{\sigma(r)}$$

where E is an effective divisor such that $\sup(E) = F \cup F_1 \cup F_2 \cup F_3 \cup L_1 \cup L_2$, and $\{C_j\}$ are the irreducible components of the exceptional locus of $Z \to \operatorname{spec}(R)$ not contained in $\sup(E)$. Let $b_i = b_i(\mathscr{C})$ and $v = v(\mathscr{C})$ be the associated constants of $\mathscr{C}$ as defined in Definition 5.

Let Δ be an effective Cartier divisor on Z with exceptional support such that $(\Delta \cdot F) = (\Delta \cdot F_1) = (\Delta \cdot F_2) = (\Delta \cdot F_3) = (\Delta \cdot L_1) = (\Delta \cdot L_2) = 0$, and

$$\deg_{C_j} \mathcal{O}_{C_j}(-\Delta) > \left(\max_{\sigma(i)=j} b_i \right) + v \tag{11}$$

for all j.

Let $\mathcal{L} = \mathcal{O}_Z(D - \Delta)$. There exists $b > 0$ such that $bW > E$. By replacing H by a sufficiently large multiple of H we may assume that $E > B$. We will show that the decomposition sequence $\mathscr{C}$ with its associated constants b_i and v, and $\mathcal{L}$ satisfy assumptions (*1*) through (*4*) of Lemma 4. Assumption (*2*) follows from (11) and the fact that D is numerically effective. Let $\mathcal{N} = \mathcal{L}$ or $\mathcal{L}^{\otimes 2}$. We will show that

$$H^1(\mathcal{O}_{bW} \otimes \mathcal{N} \otimes_Z (-mH)) = 0 \quad \text{for} \quad m \geqq 0 \tag{12}$$

and the natural map

$$H^0(\mathcal{O}_{bW} \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH)) \to H^0(\mathcal{O}_B \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH)) \tag{13}$$

is a surjection for $m \geqq 0$. For $0 \leqq a \leqq b$ and $0 \leqq n \leqq t-1$ there are exact sequences:

$$0 \to [\mathcal{O}_Z(-B-nF)/\mathcal{O}_Z(-B-(n+1)F)] \otimes \mathcal{O}_Z(-aW) \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH) \to$$
$$\mathcal{O}_{B+(n+1)F+aW} \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH) \to$$
$$\mathcal{O}_{B+nF+aW} \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH) \to 0 \tag{14}$$

$$0 \to \mathcal{O}_B(-aW) \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH) \to \mathcal{O}_{B+aW} \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH) \to$$
$$\mathcal{O}_{aW} \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH) \to 0$$
$$[\mathcal{O}_Z(-B-nF)/\mathcal{O}_Z(-B-(n+1)F)] \otimes \mathcal{O}_Z(-aW) \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH)$$
$$\simeq [\mathcal{O}_Z(-(n+2)F)/\mathcal{O}_Z(-(n+3)F)] \otimes \mathcal{O}_F(-F_1-F_2-F_3-L_1-L_2)$$
$$\otimes \mathcal{O}_Z(-aW-mH) \otimes \mathcal{N}$$
$$\simeq [\mathcal{O}_Z(-(n+2)F)/\mathcal{O}_Z(-(n+3)F))] \otimes \mathcal{O}_{\mathbf{P}^1}(-5) \otimes \mathcal{O}_Z(-aW-mH) \otimes \mathcal{N}$$

There exist $\lambda \geqq 3(n+2)-5 \geqq 1$ such that there is an inclusion

$$\mathcal{O}_{\mathbf{P}^1}(\lambda) \simeq [\mathcal{O}_Z(-F)/\mathcal{O}_Z(-2F))]^{\otimes(n+2)} \otimes \mathcal{O}_{\mathbf{P}^1}(-5) \otimes \mathcal{O}_Z(-aW-mH) \otimes \mathcal{N} \to$$
$$[\mathcal{O}_Z(-(n+2)F)/\mathcal{O}_Z(-(n+3)F)] \otimes \mathcal{O}_{\mathbf{P}^1}(-5) \otimes \mathcal{O}_Z(-aW-mH) \otimes \mathcal{N}$$

Hence $H^1([\mathcal{O}_Z(-B-nF)/\mathcal{O}_Z(-B-(n+1)F)] \otimes \mathcal{O}_Z(-aW-mH) \otimes \mathcal{N}) = 0$. $H^1(\mathcal{O}_B \otimes \mathcal{O}_Z(-aW-mH) \otimes \mathcal{N}) = 0$ since $\mathcal{O}_B \otimes \mathcal{O}_Z(-aW-mH) \otimes \mathcal{N}$ is numerically effective and $\deg_{L_i}(\mathcal{O}_Z(-aW-mH) \otimes \mathcal{N}) \geqq 1$ for $i = 1$ and 2. (14) now shows that formulas (12) and (13) are true.

There is an exact sequence

$$0 \to K \to \mathcal{O}_{bW} \otimes \mathcal{N} \to \mathcal{O}_E \otimes \mathcal{N} \to 0$$

where K is defined to be the kernel. This induces an exact sequence

$$0 = H^1(\mathcal{O}_{bW} \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH)) \to H^1(\mathcal{O}_E \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH)) \to$$
$$H^2(K \otimes \mathcal{O}_Z(-mH)) = 0$$

Hence $H^1(\mathcal{O}_E \otimes \mathcal{N} \otimes \mathcal{O}_Z(-mH)) = 0$ for all $m \geqq 0$, and assumption (*2*) of Lemma 4 is verified.

Since the composition

$$H^0(\mathcal{O}_{bW} \otimes \mathcal{N}) \to H^0(\mathcal{O}_E \otimes \mathcal{N}) \to H^0(\mathcal{O}_B \otimes \mathcal{N})$$

is surjective,

$$H^0(\mathcal{O}_E \otimes \mathcal{N}) \to H^0(\mathcal{O}_B \otimes \mathcal{N}) \text{ is surjective,} \tag{15}$$

and $\mathcal{O}_E \otimes \mathcal{L}$ is generated by global sections since $H^0(\mathcal{O}_B \otimes \mathcal{L})$ is generated by global sections. This verifies (1), (2) and (3) of the assumptions of Lemma 4.

It remains to verify assumption (*4*).

$$\deg_{L_1}(\mathcal{L}) = \deg_{L_2}(\mathcal{L}) = 1, \quad \text{and} \quad \deg_F(\mathcal{L}) = \deg_{F_i}(\mathcal{L}) = 0$$

for all i and the fact that $\mathcal{L} \otimes \mathcal{O}_B$ is generated by global sections implies that $\mathcal{L} \otimes \mathcal{O}_B = \mathcal{O}_B(P+Q)$ where P and Q are points of L_1 and L_2 respectively, which are nonsingular points of B. We know that $H^0(\mathcal{L} \otimes \mathcal{O}_B)$ is base point free. We will

show that it does not separate P and Q. Consider the exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{O}_B(P) \to \mathcal{O}_B(P)/m_P\mathcal{O}_B(P) \to 0 \tag{16}$$

where m_P is the ideal sheaf of P. This induces an exact sequence

$$H^0(\mathcal{O}_B(P)) \to \mathcal{O}_B(P)/m_P\mathcal{O}_B(P) \simeq k \to H^1(\mathcal{O}_B) \simeq k \to H^1(\mathcal{O}_B(P)) = 0$$

which shows that $H^0(\mathcal{L} \otimes \mathcal{O}_B)$ does not separate P and Q. But tensoring (16) with $\mathcal{L}$ and taking global sections shows that $\mathcal{L}^{\otimes 2}$ does separate P and Q. Hence $H^0(\mathcal{L} \otimes \mathcal{O}_B) \otimes H^0(\mathcal{L} \otimes \mathcal{O}_B) \to H^0(\mathcal{L}^{\otimes 2} \otimes \mathcal{O}_B)$ is not surjective. By (15), $H^0(\mathcal{L} \otimes \mathcal{O}_E) \otimes H^0(\mathcal{L} \otimes \mathcal{O}_E) \to H^0(\mathcal{L}^{\otimes 2} \otimes \mathcal{O}_E)$ is not surjective.

Now by Lemma 4, $H^0(\mathcal{L}) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2})$ is not surjective. The existence of such an $\mathcal{L}$ gives a contradiction by Lemma 1.

Proposition 3. *Suppose that R satisfies condition (*), $k = R/m$ is algebraically closed and that $f: X \to \operatorname{spec}(R)$ is a resolution of singularities such that f factors through Y and that Y has rational singularities, where h and g are birational projective morphisms.*

$$X \xrightarrow{h} Y \xrightarrow{g} \operatorname{spec}(R)$$

Let F be an integral curve which is exceptional for g. Let E be the proper transform of F by h. Suppose that $\operatorname{sing}(Y) \subset F$ and that the singular locus of each exceptional curve for g is contained in F. Then $\deg((I_F\mathcal{O}_X) \otimes \mathcal{O}_E) \geqq -1$.

Proof. By Propositions 1 and 2, $E \simeq F \simeq \mathbf{P}^1$. By Proposition 2 of this paper and Lemma 4 of [C1],

$$\begin{aligned} -1 &\leqq \deg_F(\mathcal{O}_Y(-F)/\mathcal{O}_Y(-2F)) = \deg_F(I_F/I_F^{(2)}) \\ &= \deg_E((I_F\mathcal{O}_X) \otimes \mathcal{O}_E)\,. \end{aligned}$$

Proposition 4. *Suppose that R satisfies condition (*), $k = R/m$ is algebraically closed, $f: X \to \operatorname{spec}(R)$ is a resolution of singularities which factors through Y, and that Y has rational singularities, where h and g are birational projective morphisms.*

$$X \xrightarrow{h} Y \xrightarrow{g} \operatorname{spec}(R)$$

Let F be an integral curve which is exceptional for g. Let E be the proper transform of F by h. Suppose that $\operatorname{sing}(Y) \subset F$ and that the singular locus of each integral exceptional curve for g is contained in F. Then $\deg((I_F^{(n)}\mathcal{O}_X) \otimes \mathcal{O}_E) \geqq -1$ for all $n > 0$.

Proof. The proof of Proposition 3 [C1] with references to "Proposition 3" of [C1] changed to "Proposition 2" of this paper, proves this proposition.

Proposition 5. *Suppose that (R, m) satisfies condition (*) and $k = R/m$ is algebraically closed. Then R has rational singularities.*

Proof of Proposition 5. Let $X \to \operatorname{spec}(R)$ be a resolution of singularities such that every integral exceptional divisor is nonsingular. There exists a factorization of birational projective morphisms

$$X = X_n \xrightarrow{h_n} X_{n-1} \xrightarrow{h_{n-1}} \cdots \longrightarrow X_1 \xrightarrow{h_1} X_0 = \operatorname{spec}(R)$$

Where each h_r contracts a single integral exceptional curve E_r, X_r is normal, and the singular and fundamental locus of X_r is contained in E_r by Theorem 2 of [C2]. Let $a^r: X \to X_r$ be the induced maps.

We will show that $R^1 a^{r-1}_* \mathcal{O}_X = 0$ for all r by induction. For $r = n+1$ it is trivial so we will suppose that $R^1 a^r_* \mathcal{O}_X = 0$, and show that $R^1 a^{r-1}_* \mathcal{O}_X = 0$. Let E'_r be the proper transform of E_r by a^r. X_r has rational singularities, so that $\deg_{E'_r}((I^{(s)}_{E'_r}\mathcal{O}_X) \otimes \mathcal{O}_{E'_r}) \geqq -1$ for all $s \geqq 0$ by Proposition 4. By Lemma 5 [C1] X_{r-1} has rational singularities.

Theorem 1. *Suppose that (R, m) is an excellent and normal local domain of dimension 2. Suppose that $k = R/m$ is algebraically closed. Then the following are equivalent.*

(1) R has a rational singularity.
(2) If I and J are any two complete ideals in R, then the product IJ is a complete ideal.
(3) If I and J are any two complete m-primary ideals in R, then the product IJ is a complete ideal.
(4) If I is any m-primary complete ideal in R, then I^2 is a complete ideal.

Proof. (1) implies (2) is Theorem 7.1 of [L]. (2) implies (3) and (3) implies (4) are obvious.

It remains to show that (4) implies (1). Let $\hat{R}$ be the completion of R with respect to m. Then the $\hat{m}$-primary ideals of $\hat{R}$ are the extensions of m-primary ideals of R. By Lemma 2, we may assume that R is complete. Theorem 1 then follows from Proposition 5.

II. It is necessary that k is algebraically closed

In this section we give a proof of Theorem 2.

Theorem 2. *Let k be a field of characteristic not equal to 3. Let $R(k)$ be the ring*

$$R(k) = k[[x, y, z]]/x^3 + 3y^3 + 9z^3 .$$

The following is true

(1) $R(k)$ is a normal local domain of dimension two and a simple elliptic singularity. In particular, it is not rational.
(2) Let $k = \mathbf{Q}$ be the rational numbers. If $I, J \subset R(\mathbf{Q})$ are any two complete ideals, then IJ is complete.
(3) If k has positive characteristic, or if k is an algebraically closed field, then there exists an m-primary complete ideal $I \subset R(k)$ such that I^2 is not complete.

Let $C = \text{proj}(k[x, y, z]/x^3 + 3y^3 + 9z^3)$. C has a natural very ample divisor $\mathcal{O}_C(1)$. C is an elliptic curve.

We will first establish (1) of Theorem 2. By the Jacobian criterion it can be seen that $R(k)$ is a normal domain. Let $\lambda: X \to \text{spec}(R(k))$ be the blow up of the maximal ideal. By direct calculation (V.2.11.4 [H]) it can be seen that λ is a resolution of singularities, the exceptional locus of λ is C, $\mathcal{O}_X(-C) \otimes \mathcal{O}_C \simeq \mathcal{O}_C(1)$, and $H^1(\mathcal{O}_X) = k$, so that $R(k)$ is a simple elliptic singularity.

Now we will establish 2) of theorem 2. Assume that $k = \mathbf{Q}$, so that $C = \text{proj}(\mathbf{Q}[x, y, z]/x^3 + 3y^3 + 9z^3)$.

Lemma 6. *The image of* $\deg: \mathrm{Pic}(C) \to \mathbf{Z}$ *is* $3\mathbf{Z}$.

Proof. Suppose that C has a line bundle L of degree 1. Then by The Riemann-Roch theorem, $h^0(L) = 1$ since C is an elliptic curve. Hence $L \simeq \mathcal{O}_C(p)$ where p is a rational point of C. This means that there is an integral solution in x, y, z, other than $x = y = z = 0$, to the diophantine equation

$$x^3 + 3y^3 + 9z^3 = 0 .$$

By the fundamental theorem of arithmetic this is impossible, since 3 must divide any solution x, y, z infinitely many times. Hence C does not have a line bundle of degree 1.

Suppose that L is a line bundle of degree 2. Then $\deg(\mathcal{O}_C(1) \otimes L^{-1}) = 1$, which contradicts the above paragraph.

Proposition 6. *Suppose that* L *and* M *are line bundles on* C *such that* L *and* M *are generated by global sections. Then* $H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$ *is surjective.*

Proof. We may assume that L and M are not isomorphic to $\mathcal{O}_C$, and that $\deg(L) \geqq \deg(M)$. First assume that L is not isomorphic to M. Then $h^1(M^{-1} \otimes L) = h^1(L) = 0$ by the Riemann-Roch theorem. Let $\sigma \in H^0(M)$. $H^0(M)$ is generated by global sections, hence there is a surjection $(\mathcal{O}_C)^t \to M$. Define F by the exact sequence

$$0 \to M^{-1} \xrightarrow{\sigma} \mathcal{O}_C \to F \to 0$$

$$\begin{array}{ccccccc}
0 \to M^{-1} \otimes L \otimes \mathcal{O}_C^t & \to & L \otimes \mathcal{O}_C^t & \longrightarrow & F \otimes L \otimes \mathcal{O}_C^t & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow L & \xrightarrow{\sigma} & L \otimes M & \longrightarrow & F \otimes L \otimes M & \to 0 &
\end{array}$$

This induces an exact sequence

$$\begin{array}{ccccccccc}
0 \to H^0(M^{-1} \otimes L) \otimes H^0(M) & \to & H^0(L) \otimes H^0(M) & \to & H^0(F \otimes L) \otimes H^0(M) & \to & 0 & & \\
\downarrow \psi_1 & & \downarrow \psi_2 & & \downarrow \psi_3 & & & & \\
0 \longrightarrow H^0(L) & \xrightarrow{\sigma} & H^0(L \otimes M) & \longrightarrow & H^0(F \otimes L \otimes M) & \to & 0 & &
\end{array}$$

ψ_3 is surjective since the kernel of the surjection $(\mathcal{O}_C)^t \otimes L \otimes F \to M \otimes L \otimes F$ has finite support.

Clearly $\sigma H^0(L)$ is contained in the image of ψ_2. Hence ψ_2 is surjective.

Now assume that $L \simeq M$. First suppose that $L \simeq \mathcal{O}_C(r)$. Then $H^0(\mathcal{O}_{\mathbf{P}^2}(1))^{\otimes i} \to H^0(\mathcal{O}_{\mathbf{P}^2}(i))$ is a surjection for all $i \geqq 0$, and $H^0(\mathcal{O}_{\mathbf{P}^2}(i)) \to H^0(\mathcal{O}_C(i))$ is a surjection for all $i \geqq 0$. Hence $H^0(L) \otimes H^0(M) \to H^0(L \otimes M)$ is a surjection.

The last case is when $M \simeq L \simeq \mathcal{O}_C(r) \otimes N$ where N is a divisor of degree 3 on C, and N is not isomorphic to $\mathcal{O}_C(1)$. First suppose that $r > 0$. By the previous analysis, we know that $H^0(M) \otimes H^0(\mathcal{O}_C(r)) \to H^0(M \otimes \mathcal{O}_C(r))$ and $H^0(M \otimes \mathcal{O}_C(r)) \otimes H^0(N) \to H^0(M \otimes \mathcal{O}_C(r) \otimes N)$ are surjective. Hence $H^0(M) \otimes H^0(\otimes_C(r)) \otimes H^0(N) \to H^0(M) \otimes H^0(L) \to H^0(M \otimes L)$ is surjective.

Suppose that $r = 0$. Then since M has degree 3, the Riemann-Roch theorem shows that there is an embedding $\varphi: C \to \mathbf{P}^2$ such that $\varphi^* \mathcal{O}(1) \simeq M \simeq N$. Hence $H^0(M) \otimes H^0(N) \to H^0(M \otimes N)$ is surjective.

Lemma 7. *Suppose that $\mathcal{L}$ is a line bundle generated by global sections on X. Then either $\deg_C(\mathcal{L} \otimes \mathcal{O}_C) = 3r$ with $r \geqq 1$ or $\mathcal{L} \simeq \mathcal{O}_X$.*

Proof. $\deg_C(\mathcal{L} \otimes \mathcal{O}_C) = 3r$ with $r \geqq 0$. For all $n \geqq 1$,

$$0 \to \mathcal{O}_C(-nC) \to \mathcal{O}^*_{(n+1)C} \to \mathcal{O}^*_{nC} \to 0$$

is exact. $h^1(\mathcal{O}_C(-nC)) = 0$ for all $n \geqq 1$. Hence $\mathrm{Pic}(X) = \varprojlim \mathrm{Pic}(nC) = \varprojlim H^1(\mathcal{O}^*_{nC}) = H^1(\mathcal{O}^*_C)$. If $\mathcal{L}$ is generated by global sections, and if $\deg(\mathcal{L}) = 0$, then $\mathcal{L} \otimes \mathcal{O}_C \simeq \mathcal{O}_C$. But then $\mathcal{L} \simeq \mathcal{O}_X$.

Lemma 8. *Suppose that $\mathcal{L}$ is a line bundle on X and that $\deg_C(\mathcal{L}) > 0$. Then $h^1(\mathcal{L}) = 0$.*

Proof. Consider the exact sequences

$$0 \to \mathcal{O}_C \otimes \mathcal{L} \otimes \mathcal{O}_X(-nC) \to \mathcal{O}_{(n+1)C} \otimes \mathcal{L} \to \mathcal{O}_{nC} \otimes \mathcal{L} \to 0$$

to see that $h^1(\mathcal{O}_{nC} \otimes \mathcal{L}) = 0$ for all $n \geqq 1$. Hence $H^1(\mathcal{L}) = \varprojlim H^1(\mathcal{O}_{nC} \otimes \mathcal{L}) = 0$.

Proposition 7. *Suppose that $\mathcal{L}$ and $\mathcal{N}$ are line bundles on X such that $\mathcal{L}$ and $\mathcal{N}$ are generated by global sections. Then $H^0(\mathcal{L}) \otimes H^0(\mathcal{N}) \to H^0(\mathcal{L} \otimes \mathcal{N})$ is surjective.*

Proof. By induction, we will show that

$$H^0(\mathcal{N}) \otimes H^0(\mathcal{L} \otimes \mathcal{O}_{nC}) \to H^0(\mathcal{N} \otimes \mathcal{L} \otimes \mathcal{O}_{nC}) \text{ is surjective} \tag{17}$$

for all $n \geqq 1$, from which the theorem follows. We will first show that $H^0(\mathcal{N}) \otimes H^0(\mathcal{L} \otimes \mathcal{O}_C) \to H^0(\mathcal{N} \otimes \mathcal{L} \otimes \mathcal{O}_C)$ is a surjection. Exactness of

$$0 \to \mathcal{O}_X(-C) \otimes \mathcal{N} \to \mathcal{N} \to \mathcal{N} \otimes \mathcal{O}_C \to 0$$

implies that $H^0(\mathcal{N}) \to H^0(\mathcal{N} \otimes \mathcal{O}_C)$ is a surjection by Lemma 8. Now the surjectivity follows from Proposition 6.

Assume that the map (17) is a surjection for some $n \geqq 1$. We will show that it is a surjection for $n + 1$. There exists a surjection $R^t \to H^0(\mathcal{N})$ of R modules. There is a diagram:

$$\begin{array}{ccccccc}
0 \to \mathcal{O}_C(-nC) \otimes \mathcal{L} \otimes \mathcal{O}_X^t & \to & \mathcal{O}_{(n+1)C} \otimes \mathcal{L} \otimes \mathcal{O}_X^t & \to & \mathcal{O}_{nC} \otimes \mathcal{L} \otimes \mathcal{O}_X^t & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to \mathcal{O}_C(-nC) \otimes \mathcal{L} \otimes \mathcal{N} & \to & \mathcal{O}_{(n+1)C} \otimes \mathcal{L} \otimes \mathcal{N} & \to & \mathcal{O}_{nC} \otimes \mathcal{L} \otimes \mathcal{N} & \to & 0
\end{array}$$

Taking global sections yields:

$$\begin{array}{ccccccc}
0 \to & H^0(\mathcal{O}_C(-nC) \otimes \mathcal{L}) \otimes R^t & \to & H^0(\mathcal{O}_{(n+1)C} \otimes \mathcal{L}) \otimes R^t & \to & H^0(\mathcal{O}_{nC} \otimes \mathcal{L}) \otimes R^t & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& H^0(\mathcal{O}_C(-nC) \otimes \mathcal{L}) \otimes H^0(\mathcal{N}) & & H^0(\mathcal{O}_{(n+1)C} \otimes \mathcal{L}) \otimes H^0(\mathcal{N}) & & H^0(\mathcal{O}_{nC} \otimes \mathcal{L}) \otimes H^0(\mathcal{N}) & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(\mathcal{O}_C(-nC) \otimes \mathcal{L} \otimes \mathcal{N}) & \to & H^0(\mathcal{O}_{(n+1)C} \otimes \mathcal{L} \otimes \mathcal{N}) & \to & H^0(\mathcal{O}_{nC} \otimes \mathcal{L} \otimes \mathcal{N}) & \to 0
\end{array}$$

The right and left two vertical maps are surjective by induction, hence the middle map is a surjection by the five lemma.

Suppose that $h: Z \to \text{spec}(R)$ is projective, birational, and Z is nonsingular. Since $X \to \text{spec}(R)$ is the minimal resolution of singularities, there exists a projective map $\alpha: Z \to X$ factoring h. Let C' be the proper transform of C by α. By Theorem 2 of [C1], there exists a projective birational map $f: Z \to Y$ where Y is normal and the exceptional locus of f is C'. Let $g: Y \to \text{spec}(R)$ be the induced map.

Lemma 9. *(1) Suppose that $\mathcal{N}$ is a line bundle on Z such that $\mathcal{N} \otimes \mathcal{O}_{C'} \simeq \mathcal{O}_{C'}$. Then $\mathcal{N} \simeq f^*(\mathcal{H})$ for some line bundle $\mathcal{H}$ on Y. (2) $H^1(\mathcal{O}_Y) = 0$.*

Proof of (1). Let $p = f(C')$, the singular point of Y. Let $Z' = Z \otimes \hat{\mathcal{O}}_{Y,p}$ where $\hat{}$ denotes completion along p. By consideration of

$$0 \to \mathcal{O}_{C'}(-nC') \to \mathcal{O}^*_{(n+1)C'} \to \mathcal{O}^*_{nC'} \to 0$$

we have that $\text{Pic}(Z') \simeq \text{Pic}(C')$. $\mathcal{N} \otimes \mathcal{O}_{C'} \simeq \mathcal{O}_{C'}$ implies that $\mathcal{N} \simeq f^*(\mathcal{H})$ for some line bundle $\mathcal{H}$ on Y by Lemma 5 of [C2].

Proof of (2). $(-C' \cdot C') \geqq 3$ and C' an elliptic curve shows that $R^1 f_* \mathcal{O}_{C'}(-nC') = 0$ for all $n > 0$. The exact sequences

$$0 \to \mathcal{O}_{C'}(-(n+1)C') \to \mathcal{O}_{(n+1)C'}(-C') \to \mathcal{O}_{nC'}(-C') \to 0$$

imply that $R^1 f_* \mathcal{O}_{nC'}(-C') = 0$ for all $n > 0$. Hence $(R^1 f_* \mathcal{O}_Z(-C')) \otimes \hat{\mathcal{O}}_{Y,p} = R^1 f_* \mathcal{O}_{Z'}(-C') = \varprojlim R^1 f_* \mathcal{O}_{nC'}(-C') = 0$. Since $\otimes \hat{\mathcal{O}}_{Y,p}$ is faithfully flat, $R^1 f_* \mathcal{O}_Z(-C') = 0$.

$$0 \to \mathcal{O}_Z(-C') \to \mathcal{O}_Z \to \mathcal{O}_{C'} \to 0$$

now shows that $R^1 f_* \mathcal{O}_Z = R^1 f_* \mathcal{O}_{C'} = H^1(\mathcal{O}_C)$. A similar argument applied to X and C shows that $R^1 \lambda_* \mathcal{O}_X = H^1(\mathcal{O}_C)$. But $R^1 h_* \mathcal{O}_Z = R^1 \lambda_* \mathcal{O}_X$ since X and Z are nonsingular. Since $R^1 f_* \mathcal{O}_Z$ has finite support, $g_* R^1 f_* \mathcal{O}_Z = H^1(\mathcal{O}_C) = R^1 h_* \mathcal{O}_Z$. By the Leray spectral sequence, the following sequence is exact.

$$0 \to R^1 g_*(\mathcal{O}_Y) \to R^1 h_*(\mathcal{O}_Z) \to g_* R^1 f_* \mathcal{O}_Z \to R^2 g_* \mathcal{O}_Y = 0\,.$$

Hence $H^1(\mathcal{O}_Y) = 0$.

Lemma 10. *Let notation be as in the paragraph preceding Lemma 9. Suppose that $\mathcal{N}$ is a line bundle on Z such that $\mathcal{N}$ is generated by global sections, and that $\mathcal{H}$ is a numerically effective line bundle on Y. Then $H^0(\mathcal{N}) \otimes H^0(\mathcal{H}) \to H^0(\mathcal{N} \otimes f^*(\mathcal{H}))$ is a surjection.*

Proof. We will use Lemma 7.3 of [L]. We need to show that $f_*(\mathcal{N})$ and $\mathcal{H}$ are generated by global sections to apply the lemma. By Theorem 12.1 of [L], and (2) of Lemma 9, $\mathcal{H}$ is generated by global sections. $f_*(\mathcal{N}) = f_*((h_*(\mathcal{N})\mathcal{O}_Y)\mathcal{O}_Z) = h_*(\mathcal{N})\mathcal{O}_Y$. But $h_*(\mathcal{N})\mathcal{O}_Y$ is complete by Proposition 6.5 [L] and (2) of Lemma 9. Hence $f_*(\mathcal{N}) = h_*(\mathcal{N})\mathcal{O}_Y$, and $f_*(\mathcal{N})$ is generated by global sections. $H^0(\mathcal{N}) \otimes H^0(\mathcal{H}) = g_*(f_*\mathcal{N}) \otimes g_*\mathcal{H}$. $H^0(\mathcal{N} \otimes f^*\mathcal{H}) = g_*(f_*(\mathcal{N} \otimes f^*\mathcal{H})) = g_*(f_*(\mathcal{N}) \otimes \mathcal{H})$ by the projection formula. By Lemma 7.3 of [L], $g_* f_*(\mathcal{N}) \otimes g_*(\mathcal{H}) \to g_*(f_*(\mathcal{N}) \otimes \mathcal{H})$ is surjective.

Proposition 8. *Suppose that $h: Z \to \text{spec}(R)$ is a resolution of singularities, and that $\mathcal{L}$ and $\mathcal{N}$ are line bundles on Z such that $\mathcal{L}$ and $\mathcal{N}$ are generated by global sections. Then $H^0(\mathcal{L}) \otimes H^0(\mathcal{N}) \to H^0(\mathcal{L} \otimes \mathcal{N})$ is a surjection.*

Proof. We will use the notation introduced before Lemma 9. Let $\mathscr{L}' = \mathscr{L} \otimes \mathcal{O}_{C'}$. $\mathscr{L}$ generated by global sections implies that $\mathscr{L}' = \mathcal{O}_{C'}$ or $\deg(\mathscr{L}') = 3r$ with $r \geqq 1$. There exists a line bundle $\mathscr{L}_3$ on X such that $\mathscr{L}_3 \otimes \mathcal{O}_C \simeq \mathscr{L}'$. $\mathscr{L}_3$ is necessarily generated by global sections. Let $\mathscr{L}_1 = \alpha^*(\mathscr{L}_3)$. Let $\mathscr{L}_2 = \mathscr{L} \otimes \mathscr{L}_1^{-1}$. Clearly $\deg_E(\mathscr{L}_2) \geqq 0$ for every exceptional curve E of α since $\mathscr{L}$ is generated by global sections and $\deg_E(\mathscr{L}_1) = 0$ unless $E = C'$. $\mathscr{L}_2 \otimes \mathcal{O}_{C'} \simeq \mathcal{O}_{C'}$ by construction, hence $\mathscr{L}_2 \simeq f^*(\mathscr{L}_4)$ where $\mathscr{L}_4$ is a numerically effective line bundle on Y by (*1*) of Lemma 9. Hence we have a decomposition $\mathscr{L} \simeq \mathscr{L}_1 \otimes \mathscr{L}_2 \simeq \alpha^*(\mathscr{L}_3) \otimes f^*(\mathscr{L}_4)$ where $\mathscr{L}_3$ is generated by global sections, and $\mathscr{L}_4$ is numerically effective. There is a similar decomposition $\mathscr{N} \simeq \mathscr{N}_1 \otimes \mathscr{N}_2 \simeq \alpha^*(\mathscr{N}_3) \otimes f^*(\mathscr{N}_4)$. We have that $H^0(\mathscr{L}_1) \otimes H^0(\mathscr{N}_1) \to H^0(\mathscr{L}_1 \otimes \mathscr{N}_1)$ is a surjection by Proposition 7. By Lemma 10, $H^0(\mathscr{L}_2) \otimes H^0(\mathscr{N}_2) \to H^0(\mathscr{L}_2 \otimes \mathscr{N}_2)$ and $H^0(\mathscr{L}_1 \otimes \mathscr{N}_1) \otimes H^0(\mathscr{L}_2 \otimes \mathscr{N}_2) \to H^0(\mathscr{L} \otimes \mathscr{N})$ are surjective. Hence $H^0(\mathscr{L}_1) \otimes H^0(\mathscr{N}_1) \otimes H^0(\mathscr{L}_2) \otimes H^0(\mathscr{N}_2) \to H^0(\mathscr{L}) \otimes H^0(\mathscr{N}) \to H^0(\mathscr{L} \otimes \mathscr{N})$ is surjective.

(*2*) of Theorem 2 follows directly from Proposition 8. Let I and J be complete ideals in R. There exists a resolution of singularities $h: Z \to \operatorname{spec}(R)$ such that $I\mathcal{O}_Z$ and $J\mathcal{O}_Z$ are line bundles which are generated by global sections. Set $\mathscr{L} = I\mathcal{O}_Z$ and $\mathscr{N} = J\mathcal{O}_Z$. Then $H^0(IJ\mathcal{O}_Z) = \overline{IJ} = IJ$ by Proposition 8.

When k is algebraically closed, (*3*) of Theorem 2 follows from Theorem 1. Assume that k has positive characteristic. Then X has a line bundle $\mathscr{L}$ of degree 2 by Corollary 5 to Theorem 2 Chap. VII Sect. 5 [W]. In fact, there exists a closed point P of degree 1 on C such that $\mathscr{L} = \mathcal{O}_C(2P)$. $\mathscr{L}$ is generated by global sections but is not very ample, and $\mathscr{L}^{\otimes 2}$ is very ample. Hence $H^0(\mathscr{L})^{\otimes 2} \to H^0(\mathscr{L}^{\otimes 2})$ is not a surjection. $\mathcal{O}_X(-C) \otimes \mathcal{O}_C(-2P) \simeq \mathcal{O}_C(Q)$ where Q is a closed point of degree 1 on C. Let $g: Y \to X$ be the blow up of Q with exceptional divisor F. Let $C' \simeq C$ be the strict transform of C by g. Let $D = g^*(-C) - F$. By construction, $\mathcal{O}_Y(D) \otimes \mathcal{O}_{C'} \simeq \mathscr{L}$. It can be checked that $\mathcal{O}_Y(D)$ is generated by global sections, and $H^0(\mathcal{O}_Y(D)) \to H^0(\mathcal{O}_{C'}(D))$, $H^0(\mathcal{O}_Y(2D)) \to H^0(\mathcal{O}_{C'}(2D))$ are surjections. $H^0(\mathcal{O}_Y(D))^{\otimes 2} \to H^0(\mathcal{O}_Y(2D))$ is not a surjection since $H^0(\mathscr{L})^{\otimes 2} \to H^0(\mathscr{L}^{\otimes 2})$ is not a surjection. Set $I = H^0(\mathcal{O}_Y(D))$. I is a complete m-primary ideal, and $I^2 \neq \overline{I^2} = H^0(\mathcal{O}_Y(2D))$.

III. An example

In this section we will construct the following example.

Example. There exists a normal but not Cohen-Macaulay scheme Z obtained by blowing up an m-primary ideal I in $R = \mathbf{C}[[x, y, z]]$ with natural map $h: Z \to \operatorname{spec}(R)$ such that $R^1 h_* \mathcal{O}_Z = 0$ and $R^2 h_* \mathcal{O}_Z \neq 0$. $\bigoplus_{n \geqq 0} I^n$ is normal but not S_3.

We will first need some simple results on projective surfaces. Let $\mathbf{P}^2$ be complex projective space. Let W be a general quartic in $\mathbf{P}^2$. Let C'' be a nonsingular cubic curve in $\mathbf{P}^2$. Let H' be a general linear hyperplane section on $\mathbf{P}^2$. $W \cdot C'' = P_1 + \ldots + P_{12}$, twelve points on C''. Let $\lambda: E' \to \mathbf{P}^2$ be the blow up of $P_1, \ldots, P_{12}$ with exceptional curves $e_1, \ldots, e_{12}$. Let $h = \lambda^*(H')$, and let C' be the proper transform of C'' by λ. C' is linearly equivalent to $3h - e_1 - \ldots - e_{12}$. Let $D = h + C'$.

Lemma 11. *Let notation be as in the above paragraph. Then $|mD|$ is base point free for all $m > 0$. There exists $n_0 > 0$ such that for all $m > 0$, $|mn_0D|$ induces a map φ on E' such that φ is an isomorphism away from C', and φ contracts C' to a normal point.*

Proof. By construction, $(D \cdot C) > 0$ for any irreducible curve C on E' except for C'.

$$(D \cdot C') = 3 + (C')^2 = 0 .$$

For all $m > 0$ $Bs|mD|$, the base locus of $|mD|$, is contained in C'. Consider the exact sequences

$$0 \to \mathcal{O}_{E'}(mD - C') \to \mathcal{O}_{E'}(mD) \to \mathcal{O}_{C'}(mD) \to 0 \tag{18}$$

We will show that $H^1(\mathcal{O}_{E'}(mD - C')) = 0$ if $m \geqq 1$. The first step is to observe that $H^1(\mathcal{O}_{E'}(rh)) = 0$ for all $r \geqq 0$. We have that $(h \cdot C') = 3$ and $(C' \cdot C') = -3$. Consider

$$0 \to \mathcal{O}_{E'}(rh + (m-1)C') \to \mathcal{O}_{E'}(rh + mC') \to \mathcal{O}_{C'}(rh + mC') \to 0 \tag{19}$$

$((rh + mC') \cdot C') = 3(r - m)$ implies that $H^1(\mathcal{O}_{C'}(rh + mC')) = 0$ if $r > m$, since C' is an elliptic curve. (19) then shows that $H^1(\mathcal{O}_{E'}(rH + mC')) = 0$ if $r > m \geqq 0$. In particular, $H^1(\mathcal{O}_{E'}(mh + (m-1)C')) = H^1(\mathcal{O}_{E'}(mD - C')) = 0$.

$\mathcal{O}_{C'}(D) \simeq \mathcal{O}_{C'}$ by construction. Hence in (18), $H^0(\mathcal{O}_{E'}(mD)) \to H^0(\mathcal{O}_{C'})$ is a surjection, so that $Bs|mD| = \varnothing$ for $m \geqq 1$.

Since $(C \cdot D) > 0$ for every curve C on E' except for C', and $(C' \cdot D) = 0$, we have that C' is the only exceptional curve for $|mD|$. The existence of an n_0 with the desired property follows from pp. 581–582 [Z].

Now we can construct the example. Let $Y = \operatorname{spec}(R)$. Let $f_2 : X_0 \to Y$ be the blow up of $m = (x, y, z)$ with exceptional surface $E_0' \simeq \mathbf{P}^2$. We will use the notation preceding Lemma 11. Let $f_1 : X \to X_0$ be the blow up of $W \cdot C'' = P_1 + \ldots + P_{12}$ with exceptional surfaces $E_1, \ldots, E_{12}$. Let $f : X \to Y$ be the natural map. Let E' be the proper transform of E_0'. E' is the surface constructed in the paragraph preceding Lemma 11. Let $H = -E' - E_1 - \ldots - E_{12}$ so that $\mathcal{O}_X(H) = m\mathcal{O}_X$. $K_X = -2H + 2E_1 + \ldots + 2E_{12}$ is the canonical divisor of X. Let $D' = 4H - E_1 - \ldots - E_{12}$. $D' \cdot E' = D$, the divisor constructed in the paragraph preceding Lemma 11. For each i, $\mathcal{O}_{E_i} \otimes \mathcal{O}_X(D') \simeq \mathcal{O}_{\mathbf{P}^2}(1)$.

Suppose that an irreducible curve C is supported in $E' + E_1 + \ldots + E_{12}$. Then $(C \cdot D') > 0$ unless $C = C'$, and $(C' \cdot D') = 0$.

$$-E' = H + E_1 + \ldots + E_{12}$$

$$-2E' - K_X = 2H + 2E_1 + \ldots + 2E_{12} + 2H - 2E_1 - \ldots - 2E_{12} = 4H$$

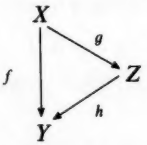
Hence $\mathcal{O}_X(mD' - 2E') \simeq \mathcal{O}_X(mD' + 4H + K_X)$. For $m \geqq 0$, $H^1(\mathcal{O}_X(mD' - 2E')) = 0$ by the Kawamata-Viehweg Vanishing Theorem, Theorem 1.2 [KMM]. Hence there is a surjection $H^0(\mathcal{O}_X(mD')) \to H^0(\mathcal{O}_{2E'}(mD'))$. There is an exact sequence

$$0 \to \mathcal{O}_{E'}(mD' - E') \to \mathcal{O}_{2E'}(mD') \to \mathcal{O}_{E'}(mD') \to 0$$

By the adjunction formula, $\mathcal{O}_{E'}(K_{E'}) = \mathcal{O}_{E'}(K_X + E')$. Hence $\mathcal{O}_{E'}(mD' - E') = \mathcal{O}_{E'}(mD' + 4H + K_{E'})$. The Kawamata-Viehweg Vanishing Theorem implies that $H^1(\mathcal{O}_{E'}(mD - E')) = 0$. Hence $H^0(\mathcal{O}_X(mD')) \to H^0(\mathcal{O}_{E'}(mD))$ is a surjection. $H^0(\mathcal{O}_{E'}(mD))$ is generated by global sections by Lemma 11. Hence $Bs|mD| \subset \cup (E_i - e_i)$ which implies that $Bs|mD|$ is a finite number of points. By a simple modification of Theorem 6.1 [Z2], there exists $m_0 > 0$ such that

$Bs|m_0D| = \emptyset$. Let $A = \bigoplus_{m \geq 0} f_* \mathcal{O}_X(mD)$. A is normal and a finitely generated R algebra.

Let $Z = \text{Proj}(A)$. Z is the normalization of the image of $|m_0D|$. Let g and h be the natural maps defined by the following diagram.



There is an exact sequence obtained from the Leray spectral sequence

$$0 \to R^1 h_* \mathcal{O}_Z \to R^1 f_* \mathcal{O}_X \to h_* R^1 g_* \mathcal{O}_X \to R^2 h_* \mathcal{O}_Z \to R^2 f_* \mathcal{O}_X \tag{20}$$

$R^1 f_* \mathcal{O}_X = R^2 f_* \mathcal{O}_X = 0$ since X and Y are nonsingular. g is an isomorphism away from C', and C' is contracted to a normal point Q on Z. Hence $R^1 g_* \mathcal{O}_X$ is supported at Q. C' is an elliptic curve so that $R^1 g_* \mathcal{O}_{C'} = H^1(\mathcal{O}_{C'}) = \mathbf{C}$. The exactness of

$$R^1 g_* \mathcal{O}_X \to R^1 g_* \mathcal{O}_{C'} \to R^2 g_* \mathcal{O}_X(-C')$$

and the fact that $R^2 g_* \mathcal{O}_X(-C') = 0$ since fibers of g have dimension less than 2, shows that $R^1 g_* \mathcal{O}_X \neq 0$. Since $R^1 g_* \mathcal{O}_X$ is supported at the point Q, $h_* R^1 g_* \mathcal{O}_X \neq 0$. Examination of (20) now shows that $R^2 h_* \mathcal{O}_Z \neq 0$, and $R^1 h_* \mathcal{O}_Z = 0$.

Z is nonsingular except at the point Q. Z is not Cohen-Macaulay at Q by 2.6.1 [B] or Lemma 0.5 [BeSo] since $g: X \to Z$ is a resolution of singularities, and $R^1 g_* \mathcal{O}_X$ is supported at Q.

The scheme Z constructed above is obtained by blowing up an m-primary ideal J. Hence for some $r \gg 0$, if $I = \overline{J^r}$, $\bigoplus_{n \geq 0} I^n$ is normal and $Z = \text{proj}(\bigoplus_{n \geq 0} I^n)$. Z is not Cohen-Macaulay implies that $\bigoplus_{n \geq 0} I^n$ is not S_3.

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Inventiones mathematicae

Gaussian kernels have only Gaussian maximizers*

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Abstract. A Gaussian integral kernel $G(x, y)$ on $\mathbf{R}^n \times \mathbf{R}^n$ is the exponential of a quadratic form in x and y; the Fourier transform kernel is an example. The problem addressed here is to find the sharp bound of G as an operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ and to prove that the $L^p(\mathbf{R}^n)$ functions that saturate the bound are necessarily Gaussians. This is accomplished generally for $1 < p \leqq q < \infty$ and also for $p > q$ in some special cases. Besides greatly extending previous results in this area, the proof technique is also essentially different from earlier ones. A corollary of these results is a fully multidimensional, multilinear generalization of Young's inequality.

I. Introduction

The classic Hausdorff-Young-Titchmarsh [T] inequality for Fourier integrals states that for $1 \leqq p \leqq 2$ the Fourier transform on $L^p(\mathbf{R}^n)$ is a bounded map into $L^{p'}(\mathbf{R}^n)$ with a bound that is at most 1; here $1/p' + 1/p = 1$. In 1961 Babenko [BA] showed that when p' is an even integer greater than 2 and $n = 1$ the bound is in fact less than 1, and he determined its value. This bound is achieved for Gaussian functions and Babenko states, but does not demonstrate explicitly, that Gaussians are the only functions with this property. Babenko's method was to apply analytic function theory to the Euler-Lagrange equation associated with the maximization problem.

The Fourier integral is but one example of a transform given by a Gaussian integral kernel $G(x, y)$, i.e., the exponential of a quadratic plus linear form in x and y. In the Fourier transform case in $\mathbf{R}^n$ the kernel is $G(x, y) = \exp\{-2i(x, y)\}$. Another well known example in $\mathbf{R}^n$ is the purely real operator $\mathscr{T} = \exp\{t\Delta + 2tx \cdot \nabla\}$ on Gauss space (with measure $d\mu = \exp\{-|x|^2\}\, dx$) investigated by Nelson [N1; N2] as an operator from $L^p(\mathbf{R}^n, d\mu)$ to $L^q(\mathbf{R}^n, d\mu)$. In

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terms of Lebesgue measure, this amounts to considering the kernel

$$G(x, y) = \exp\left\{ -\frac{1}{q}|x|^2 + \frac{1}{p}|y|^2 - \frac{|y - cx|^2}{(1 - c^2)} \right\}$$

from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ for $0 \leqq c = e^{-t} < 1$. Nelson defined the operator $\mathscr{G}$ by $(\mathscr{G}f)(x) = \int G(x, y) f(y)\, dy$ and showed that $\mathscr{G}$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ when $p \leqq q$ if and only if $(q - 1)c^2 \leqq p - 1$; he also derived the explicit value of the bound—which again is achieved when f is a Gaussian. This is the famous hypercontractivity theorem. [In [N1] Nelson showed that $\mathscr{G}$ is bounded if c is small enough; Glimm [GL] used this fact plus the spectral gap in the generator to show that $\tilde{\mathscr{G}}$ is a contraction on Gauss space for some still smaller c. Finally Nelson [N2] proved the sharp bound as stated above. In 1976 Neveu [NE] and Brascamp and Lieb [BL] found other proofs, and Simon [SI] found a proof for $p = 2$ and $q = 2, 4, 6, 8 \ldots$. Recently, Carlen and Loss [CL] have used their method of competing symmetries to construct another proof of the hypercontractivity theorem.] However, Nelson's method seems incapable of showing that Gaussians are the *only* maximizers; the proof of this fact, as well as a completely different proof (using rearrangement inequalities) of the hypercontractivity theorem was given by Brascamp and Lieb [BL]. The method in [CL] also yields uniqueness. Nelson's original proof used stochastic integrals and Gaussian processes in $\mathbf{R}^n$ (in fact it even extends to infinite dimensions). Segal [S] showed how to use Minkowski's inequality [HLP] to reduce the $\mathbf{R}^n$ case of Nelson's kernel to the $\mathbf{R}^1$ case; he also showed that $\tilde{\mathscr{G}}$ is a contraction on Gauss space for small c. The $\mathbf{R}^1$ case was simplified by Gross [G] who showed the equivalence of hypercontractivity with logarithmic Sobolev inequalities and built up one-dimensional Gauss measure from two-point measures via the central limit theorem. See the survey by Davies et al. [DGS].

In his important 1975 paper, Beckner [B1; B2] used the Nelson-Gross machinery and the Hermite semigroup to settle the question raised by Babenko. By using the tensor product structure of Fourier transforms and an application of Minkowski's inequality related to, but distinct from, Segal's [S], he reduced the $\mathbf{R}^n$ case to the $\mathbf{R}^1$ case. He also showed that for all $1 \leqq p \leqq 2$ the sharp constant in the Hausdorff-Young-Titchmarsh inequality is given by Gaussian functions—as found by Babenko. However, this method also leaves open the question of whether Gaussian functions are the *only* maximizers.

Since then the Nelson-Gross-Beckner method has been extended to other complex (as distinct from purely real or purely imaginary) Gaussian kernels in $\mathbf{R}^n$ (i.e., the complex Mehler kernel) [C; E; J; W]. In this paper the *general* problem in $\mathbf{R}^n$ in the $p \leqq q$ case will be settled by a *completely different method* and, moreover, *the maximizers will be shown to be Gaussian functions.* Some of the $p > q$ cases will be settled as well. Before discussing the earlier results in detail it is necessary to define the problem more completely.

The most general **Gaussian kernel** on $\mathbf{R}^n \times \mathbf{R}^n$ is

$$G(x, y) = \exp\left\{ -(x, Ax) - (y, By) - 2(x, Dy) + 2\left(L, \begin{pmatrix} x \\ y \end{pmatrix}\right) \right\} \tag{1.1}$$

and its action on complex valued, measurable functions $f: \mathbf{R}^n \to \mathbf{C}$, is formally given by

$$(\mathscr{G}f)(x) = \int G(x, y) f(y)\, dy\,. \tag{1.2}$$

In (1.1) A, B and D are (complex) $n \times n$ matrices with A and B being symmetric while L is a vector in $\mathbf{C}^{2n}$. The Fourier transform corresponds to $A = B = 0$, $L = 0$ and $D = iI$, with I denoting the identity.

Notation. If α and β are vectors in $\mathbf{C}^n$ then $(\alpha, \beta) \equiv \sum_{i=1}^n \alpha_i \beta_i$ and *not* $\sum_{i=1}^n \bar{\alpha}_i \beta_i$. Lebesgue integration over $\mathbf{R}^n$ is denoted simply by $\int dx$ whenever the n in question is clear from the context. The $L^p(\mathbf{R}^n)$ norm of a measurable function f will be denoted by $\|f\|_p$, i.e., $\{\int |f(x)|^p dx\}^{1/p}$. The notation

$$\begin{pmatrix} A & D \\ D^T & B \end{pmatrix} = M + iN \tag{1.3}$$

will also be used, where M and N are real, symmetric $2n \times 2n$ matrices. The sole condition imposed on G is that M is positive semidefinite. G is said to be **nondegenerate** if M is positive definite, while G is said to be **degenerate** if M has a zero eigenvalue. The Fourier transform kernel and Nelson's kernel with $(q-1)c^2 = (p-1)$ are examples of degenerate kernels. The operator $\mathscr{G}$ should perhaps be written $\mathscr{G}_G$, but this will not be done since the pairing of $\mathscr{G}$ and G will always be clear from the context.

The linear operator $\mathscr{G}$ associated to G will be studied as an operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ for $1 < p < \infty$ and $1 < q < \infty$. (The cases p or $q = 1$ or ∞ can also be analyzed by the methods of this paper but they will be omitted since these cases involve extra technical considerations.) When G is nondegenerate the definition of $\mathscr{G}$ in (1.2) makes sense (by Hölder's inequality) but if G is degenerate then (1.2) is meaningless unless f is also in $L^p(\mathbf{R}^n) \cap L^1(R^n)$. Assuming that $\mathscr{G}$, when restricted to $L^p(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$, is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ then, for any $f \in L^p(\mathbf{R}^n)$, $\mathscr{G}f \in L^q(\mathbf{R}^n)$ is uniquely defined by taking *any* sequence $f_j \in L^p(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ that converges to f in $L^p(\mathbf{R}^n)$ and then noting that $\mathscr{G}f = \lim_{j \to \infty} \mathscr{G}f_j$ is well defined since $\mathscr{G}f_j$ is a Cauchy sequence in $L^q(\mathbf{R}^n)$. This definition is well known and is, in fact, the way that the Fourier transform is defined when $1 < p \leqq 2$.

Associated to G and the numbers p and q with $1 \leqq p \leqq \infty$ and $1 \leqq q \leqq \infty$ is the ratio

$$\mathscr{R}_{p \to q}(f) = \frac{\|\mathscr{G}f\|_q}{\|f\|_p} \tag{1.4}$$

for $f \in L^p(\mathbf{R}^n)$, $f \neq 0$ and, in case G is degenerate, $f \in L^1(\mathbf{R}^n)$ as well. The norm of $\mathscr{G}$ from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ is defined to be

$$C_{p \to q} = \sup_f \mathscr{R}_{p \to q}(f) \tag{1.5}$$

in which the supremum is over the class of f's just stated. In case $0 \neq f \in L^p(\mathbf{R}^n)$ and $C_{p \to q} < \infty$ and

$$\|\mathscr{G}f\|_q = C_{p \to q} \|f\|_p$$

(using the above definition of $\mathscr{G}f$ as a limit when G is degenerate) then f is said to be a **maximizer** for $\mathscr{G}$ (or for G). If there is any ambiguity about the G under discussion (e.g., in Theorem 3.3) the notation $\mathscr{R}_{p \to q}(G, f)$ and $C_{p \to q}(G)$ will be used.

Functions from $\mathbf{R}^n$ to $\mathbf{C}$ of the form

$$g(x) = \mu \exp\{-(x, Jx) + (l, x)\} \tag{1.6}$$

with $0 \neq \mu \in \mathbf{C}$, $l \in \mathbf{C}^n$ and J a symmetric $n \times n$ matrix with $\mathrm{Re}(J)$ positive *definite* will be called **Gaussian functions**. In case $L = 0$ in (1.1) or $l = 0$ in (1.6) then G (resp. g) will be called a **centered Gaussian kernel** (resp. **function**). If A, B, D and L in (1.1) are real then G is said to be a **real Gaussian kernel**. Likewise, if J and l (but not necessarily μ) in (1.6) are real then g is said to be a **real Gaussian function**.

A preliminary simplification of G can be made. Without loss of generality it can be assumed that A and B are real matrices because the imaginary part of B can be absorbed into f in (1.4) without changing $\|f\|_p$. The imaginary part of A can be omitted without changing $\|\mathscr{G}f\|_q$. For the same reason the vector L can be assumed to be real. Furthermore, when G is nondegenerate then we can also set L (which is now real) equal to zero. The reason is simply that the affine change of variables $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} - V$, with V being the unique solution of the equation $MV = L$ in $\mathbf{R}^{2n}$, eliminates the real linear term from (1.1) and merely changes $C_{p\to q}$ into $C_{p\to q}\exp\{(L, V)\}$. When G is degenerate, L can also be eliminated in the same way provided $MV = L$ has a solution. Because $\mathrm{Rank}(M) < 2n$ in the degenerate case, such a solution conceivably might not exist, but it turns out that a solution does indeed exist whenever $\mathscr{G}$ is bounded. This is the content of Lemma 2.2 below. Therefore, *without loss of generality, the only G's that need to be studied are those for which*

(*i*) A and B are real, symmetric $n \times n$ matrices,
(*ii*) $L = 0$, i.e., G is centered.

These assumptions will be made in the theorems in this paper.

On the other hand, suppose that the supremum of $\mathscr{R}_{p\to q}(f)$ in (1.5) is taken over Gaussian functions *only* (which are automatically in $L^p(\mathbf{R}^n)$ for every p). Then, according to Lemma 2.3 below, only centered Gaussian functions need be considered in (1.5). This is a considerable simplification that is not altogether obvious and it is important in the application of Theorem 4.1 which states that this restricted supremum is all that need be considered.

The results of this paper can be summarized as follows. Three cases are treated. With the assumptions (i) and (ii) above,

(*A*) D is real and $1 < p < \infty$ and $1 < q < \infty$
(*B*) D is imaginary and either $1 < p \leqq 2$ and $1 < q < \infty$ or else $1 < p < \infty$, and $2 \leqq q < \infty$.
(*C*) D is complex and $1 < p \leqq q < \infty$.

If G is *nondegenerate* then $\mathscr{R}_{p\to q}$ has exactly one maximizer and it is a centered Gaussian function. These are Theorems 3.2, 3.3 and 3.4.

If G is *degenerate* then in all cases

$$C_{p\to q} = \sup_g \mathscr{R}_{p\to q}(g)\,, \tag{1.7}$$

where the supremum is over centered Gaussian functions. This is Theorem 4.1. Furthermore, if the supremum in (1.7) is achieved for some Gaussian function then, when $p < q$, *every maximizer is a Gaussian function*—as Theorems 4.5 shows. Theorem 4.3 gives a sufficient condition for the achievement of the maximum in (1.7) in the degenerate case; in Case (A) it is necessary as well. Thus, Case (A) is

settled completely: in the degenerate case with $p \geqq q$ there is no maximizer of any kind, while if $p < q$ all maximizers are Gaussian functions.

In general, the question of the existence of a maximizer in the degenerate case is a subtle one. For the Fourier transform (which is both Case (B) and (C)), *every* function in $L^2(\mathbf{R}^n)$ is a maximizer when $p = q = 2$; on the other hand the seemingly harmless modification of the Fourier transform in 4.2(5) below is bounded but has *no maximizer of any kind* when $q = p' \geqq 2$. When $q = p' > 2$ the Fourier transform on $\mathbf{R}^1$ has a three real parameter family of maximizers, $f(y) = \exp\{-Jy^2 + ly\}$ with $J > 0$ and $l \in \mathbf{C}$. When $p < q$ the convolution kernel $G(x, y) = \exp\{-(x-y)^2\}$ on $\mathbf{R}^1$ has a one real parameter family of maximizers, $f(y) = \exp\{-Jy^2 + ly\}$ with $l \in \mathbf{R}$ and $J = \dfrac{p'}{q'} - 1$; when $p = q$, G is bounded but there is no maximizer (see 4.2 below). There does not seem to be any simple rule. In simple cases (which include all the standard ones in $\mathbf{R}^n$ and all the cases in $\mathbf{R}^1$) the existence of a Gaussian maximizer in (1.7) can be decided by computation. Otherwise, (1.7) reduces to a complicated algebraic problem and precise conditions are not given here. Moreover it is not even proved that the absence of a Gaussian maximizer in (1.7) precludes the existence of a non-Gaussian maximizer—although a conjecture to this effect is made in 4.4.

All these results extend to Gaussian kernels on $\mathbf{R}^m \times \mathbf{R}^n$, in which A is $m \times m$, B is $n \times n$, D is $m \times n$ and $L \in \mathbf{C}^{m+n}$. The proof is given in Sect. V. This generalization, while it is an easy one, does occur in applications, e.g., the entropy bound for coherent states in [L1].

Multilinear Gaussian forms are discussed in Sect. VI and it is proved there that the methods and results of Sects. II–V carry through for real forms. As an application of the real multilinear result in Sect. 6.1, the *fully multidimensional Young inequality* for K functions (which was left unresolved in [BL], p. 162) is proved in 6.2. The method of proof is, of course, quite different from that in [BL]; there, rearrangement inequalities were used and they were not flexible enough to encompass the fully multidimensional case.

The relationship of the results of this paper to earlier results on Gaussian kernels (beyond [BA; N1; N2; B1; B2]) can be summarized as follows. In 1976 Brascamp and Lieb [BL] found the norm for Case (A) in $\mathbf{R}^n$ (Theorem 7) and proved that Gaussian functions are the unique maximizer in $\mathbf{R}^1$ in the degenerate case (Theorem 13); this latter proof easily extends to $\mathbf{R}^n$ and to the nondegenerate case. In fact, by a simple change of variables (see the proof of Theorem 4.3 below) the $\mathbf{R}^n$ Case (A) reduces to a simple tensor product of $\mathbf{R}^1$ kernels. In 1979 Coifman et al. [C] used Beckner's result and an interpolation technique to deduce the norm for the complex Mehler kernel in $\mathbf{R}^1$ for $q = p' \geqq 2$ (which is in Case (C)). In the same year Weissler [W] extended Nelson's and Beckner's results to the complex Mehler kernel in $\mathbf{R}^1$ with the exception of $2 < p \leqq q < 3$ and $\frac{3}{2} < p \leqq q < 2$. In 1988, Epperson [E] found the norm for the following nondegenerate cases in $\mathbf{R}^1$: Case (C), Case (B), the case $p \geqq 2 \geqq q$. He also found the norm for certain $\mathbf{R}^1$ cases $q < p < 2$ and $2 < q < p$ with sufficiently nondegenerate kernels (Theorem 2.10), and for the $\mathbf{R}^1$ degenerate Case (C) if $A > 0$ and $B > 0$ (corresponding to Theorem 4.3 here).

The only complex cases in $\mathbf{R}^n$ that were known prior to Epperson's work were the simple tensor products of $\mathbf{R}^1$ kernels; these could be analyzed for $p \leqq q$ via Minkowski's inequality, as shown by Beckner [B1; B2]. Epperson was able

to handle the nondegenerate Case (C) for which there is an $n \times n$ complex symmetric matrix W with $\|W\| \leqq 1$ such that $A = W(I - W^2)^{-1}W - \frac{1}{q}I$, $B = (I - W^2)^{-1} - \frac{1}{p'}I$ and $D = W(I - W^2)^{-1}$. Here, I is the identity matrix.

It will be seen from the above summary that all the previous cases, except for Epperson's $\mathbf{R}^1$ cases of $p \geqq 2 \geqq q$ and the special $q < p < 2$ and $2 < q < p$ cases, are covered in the cases (A), (B) and (C) treated in this paper. Moreover cases (A), (B) and (C) are resolved here in full $\mathbf{R}^n$ generality (i.e., not only for simple n-fold tensor products of $\mathbf{R}^1$ kernels). The main methodological point of this paper, however, is that all the previous results, except for [BL] and [BA], ultimately rely on the Nelson-Gross machinery which, while it is natural in its original context of quantum field theory and Gauss measures, is conceptually complicated in the context of general Gaussian kernels with Lebesgue measure. The two settings (Gauss measure and Lebesgue measure) for Gaussian kernels are mathematically equivalent, however, and the choice is a matter of taste. Lebesgue measure is used in this paper because it is felt that it is more natural to retain translation invariance (e.g., in the Fourier transform). Prior to Epperson's work all results in the field, except for [BL] and [BA] came from translating Gauss measure bounds for products of complex $\mathbf{R}^1$ Mehler kernels into $\mathbf{R}^n$ results via Beckner's Minkowski lemma. The proofs here use only Minkowski's inequality and simple facts about analytic functions (which appear to be unrelated to Babenko's use of analyticity—the Euler-Lagrange equation is not used).

Basically there is one idea that runs through Theorems 3.1, 3.3 and 4.5, although the technicalities are different in each. The main idea is to study $\mathscr{G} \otimes \mathscr{G}$ from $L^p(\mathbf{R}^{2n})$ to $L^q(\mathbf{R}^{2n})$ and use Minkowski's inequality. By considering the $\mathscr{G} \otimes \mathscr{G}$ maximizer $F(y_1, y_2) = f\left(\frac{y_1 + y_2}{\sqrt{2}}\right) f\left(\frac{y_1 - y_2}{\sqrt{2}}\right)$, where f is a maximizer for $\mathscr{G}$, it is possible to conclude that f must be a Gaussian. It will be noted that some of the proofs are long, and so it may appear at first that their structure is not really very simple. To a large extent the length is due to the fact that proving uniqueness raises technical considerations that would be absent if only inequalities are proved, e.g., it is not sufficient here to prove the inequalities for a dense set of smooth functions.

Apart from the extension to $\mathbf{R}^n$ (which is handled here in a natural way) the main new theorem in this paper is that a maximizer *must be* a Gaussian, and it is unique in the nondegenerate case. In the degenerate case $C_{p \to q}(G)$ is determined by examining only Gaussian functions and, if a Gaussian maximizer exists, every maximizer is a Gaussian. This is Theorem 4.5 and it can be useful as in [L1] and [L2]. Except for the real case [BL], it was previously known that Gaussian functions were *among* the maximizers. The one exception to this rule was pointed out by Beckner (private communication) for the Fourier transform from $L^p(\mathbf{R}^n)$ to $L^{p'}(\mathbf{R}^n)$ with the restriction $p' \geqq 4$. His proof that a maximizer must be a Gaussian function in this case uses a result in [BL]; the proof is

$$\|f\|_p^2 \geqq \mu_r \|f * f\|_r \geqq \mu_r (C_r^B)^n \|\widehat{f * f}\|_{r'} = \mu_r (C_r^B)^n \|(\hat{f})^2\|_{r'} = \mu (C_r^B)^n \|\hat{f}\|_{p'}^2$$

with $r' = p'/2 \geqq 2$, with $(C_r^B)^n$ being the sharp Beckner (or Babenko) constant for the Fourier transform (denoted by $\widehat{\ }$), and with μ_r being the sharp constant in Young's convolution inequality which was derived simultaneously in [B1, B2] and in [BL]. A Gaussian function $f(y) = \exp\{-Jy^2 + ly\}$ with $J > 0$ and $l \in \mathbf{C}$ gives

equality above. However, [BL] (Theorem 13) proved that such functions are the *only* ones that give equality in Young's inequality.

It is a pleasure to acknowledge my debt to Eric Carlen. He helped to stimulate my interest in this problem and to understand the literature in the field. He also critically examined the work as it took shape. Thanks are also due to the Institute for Advanced Study for its hospitality during part of this work, and to Michael Loss for valuable discussions.

II. Some basic properties of Gaussians

2.1. Lemma (nondegenerate Gaussian kernels are compact and have maximizers). *Let G be a centered, nondegenerate Gaussian kernel in $\mathbf{R}^n \times \mathbf{R}^n$ as in* (1.1) *with M in* (1.3) *positive definite and $L = 0$. Let $1 < p < \infty$ and $1 < q < \infty$. Then $\mathcal{G}$ in* (1.2) *is a compact operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ and there is at least one maximizer $f \in L^p(\mathbf{R}^n)$ (i.e., $\mathcal{R}_{p\to q}(f) = C_{p\to q}$).*

Every such maximizer $f: \mathbf{R}^n \to \mathbf{C}$, has the following three properties, in which α and β are positive constants that depend on G, p and q but not on f.

(a) *There is an entire analytic function of order at most 2, $m: \mathbf{C}^n \to \mathbf{C}$, such that $f(x) = |m(x)|^{p'} m(x)^{-1}$ for $x \in \mathbf{R}^n$. Here $1/p + 1/p' = 1$. Moreover, for $z \in \mathbf{C}^n$,*

$$|m(z)| \leqq \alpha \| f \|_p^{p-1} \exp\{\beta |z|^2\} .$$

(b) *The function $|f|^{2(p-1)}$ from $\mathbf{R}^n$ to $\mathbf{R}$ has an extension to an entire analytic function from $\mathbf{C}^n$ to $\mathbf{C}$ whose order is at most 2. If $g: \mathbf{C}^n \to \mathbf{C}$ is this extension then for $z \in \mathbf{C}^n$*

$$|g(z)| \leqq \alpha \| f \|_p^{2(p-1)} \exp\{\beta |z|^2\} .$$

(c) *For $x \in \mathbf{R}^n$*

$$|f(x)| \leqq \alpha \| f \|_p \exp\{ -\beta(x, x)\} .$$

Finally, if $f_j \in L^p(\mathbf{R}^n)$ for $j = 1, 2, 3, \ldots$ is an L^p bounded maximizing sequence for G (i.e., $\mathcal{R}_{p\to q}(f_j) \to C_{p\to q}$) then there is a function $f \in L^p(\mathbf{R}^n)$ and a subsequence $j(1), j(2), \ldots$ such that $f_{j(k)} \to f$ strongly in $L^p(\mathbf{R}^n)$ as $k \to \infty$. If $f \neq 0$ (i.e., if $\| f_j \|_p \to 0$ as $j \to \infty$) then f is a maximizer.

Proof. For any $f \in L^p(\mathbf{R}^n)$, Hölders inequality can be used to deduce

$$|(\mathcal{G} f)(x)| \leqq T(x) \| f \|_p \tag{1}$$

with $T(x) = \| G(x, \cdot) \|_{p'}$. Simple computation shows that there are positive numbers γ and δ depending only on G and p such that $|T(x)| \leqq \gamma \exp\{ -\delta(x, x)\}$. The fact that G is nondegenerate is crucial for this result. The fact that $T \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$ shows that $\mathcal{G}$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$. Now suppose that $f_j \in L^p(\mathbf{R}^n)$ is a sequence that converges weakly in $L^p(\mathbf{R}^n)$ to some $f \in L^p(\mathbf{R}^n)$ as $j \to \infty$. Since, for each $x \in \mathbf{R}^n$, $G(x, \cdot)$ is in $L^{p'}(\mathbf{R}^n)$, it follows that $(\mathcal{G} f_j)(x) \to (\mathcal{G} f)(x)$ as $j \to \infty$ for each $x \in \mathbf{R}^n$. It can be assumed that the f_j and f satisfy $\| f_j \|_p$ and $\| f \|_p \leqq C$ for some $C > 0$ and hence, from (1), the functions $\mathcal{G} f_j$ and $\mathcal{G} f$ are bounded pointwise by the function CT. Since $T \in L^q(\mathbf{R}^n)$, $\| \mathcal{G} f_j - \mathcal{G} f \|_q \to 0$ by dominated convergence. Thus $\mathcal{G}$ takes weakly convergent sequences in $L^p(\mathbf{R}^n)$ into strongly convergent sequences in $L^q(\mathbf{R}^n)$, and so $\mathcal{G}$ is compact.

Now let f_j be a bounded maximizing sequence, i.e., $\mathscr{R}_{p\to q}(f_j) \to C_{p\to q}$ as $j \to \infty$. We can assume $\|f_j\|_p = 1$ for each j. By the Banach-Alaoglu theorem, there is an $f \in L^p(\mathbf{R}^n)$ and a subsequence $j(1), j(2), \ldots$ such that $f_j \to f$ weakly in $L^p(\mathbf{R}^n)$. As is well known, $\|f\|_p \leqq 1$. Then, by the strong convergence proved above

$$C_{p\to q} = \lim_{k\to\infty} \|\mathscr{G}f_{j(k)}\|_q = \|\mathscr{G}f\|_q \leqq C_{p\to q}\|f\|_p \leqq C_{p\to q}\,.$$

This implies that $\|f\|_p = 1$ and that f is a maximizer. Moreover, the fact that $\|f\|_p = 1$ implies (by the uniform convexity of the L^p norm) that $f_{j(k)}$ converges to f strongly in $L^p(\mathbf{R}^n)$. Thus, the first and last assertions of the lemma have been proved.

It remains to prove that a maximizer f satisfies conditions (a), (b) and (c) and it suffices to assume that $\|f\|_p = 1$. There is a function $h \in L^{q'}(\mathbf{R}^n)$ such that $\|h\|_{q'} = 1$ and $C_{p\to q} = \|\mathscr{G}f\|_q = \int h(x)(\mathscr{G}f)(x)\mathrm{d}x$. Let

$$m(y) = \int G(x, y)h(x)\mathrm{d}x = \mathrm{e}^{-(y, By)}\int \mathrm{e}^{-(x, Ax)-2(x, Dy)}h(x)\mathrm{d}x \tag{2}$$

so that, as in the proof of (1) above, $|m(y)| \leqq W(y) \equiv \mu \exp\{-\nu(y, y)\}$ for suitable positive numbers μ and ν which depend only G and q. Hölder's inequality implies that the function $(x, y) \mapsto h(x)G(x, y)f(y)$ is in $L^1(\mathbf{R}^n \times \mathbf{R}^n)$, and Fubini's theorem then implies that $\|\mathscr{G}f\|_q = \int m(y)f(y)\mathrm{d}y$. If $m(y) \equiv |m(y)|\exp\{\mathrm{i}\theta(y)\}$, the optimum choice for f is $f(y) = [|m(y)|/\|m\|_{p'}]^{p'-1}\exp\{-\mathrm{i}\theta(y)\}$, for otherwise $\mathscr{R}_{p\to q}(f)$ can be increased.

The function $m: \mathbf{R}^n \to \mathbf{C}$ has an extension to an entire analytic function on $\mathbf{C}^n$ of order at most 2. This can be seen easily from the representation (2) above and Hölder's inequality; if $y_j = u_j + \mathrm{i}v_j$ for $j = 1, \ldots, n$ and $D = E + \mathrm{i}H$ with u_j, v_j, E and H real then

$$\begin{aligned}|m(y)| &\leqq \exp\{(v, Bv) - (u, Bu)\}\,[\int \exp\{-q'(x, Ax) - 2q'(x, Eu) + 2q'(x, Hv)\}\mathrm{d}x]^{1/q'}\\ &= (\text{const.})\exp\{(v, Bv) - (u, Bu) + (Eu - Hv, A^{-1}(Eu - Hv))\}\,.\end{aligned}$$

Thus $|m(y)| \leqq (\text{const.})\exp\{(\text{const.})[(u, u) + (v, v)]\}$ which implies that the order of m is at most 2. This establishes conclusion (a). Since m is entire, the function $y \mapsto m^*(y) \equiv \overline{m(\bar{y})}$ (with the bar denoting complex conjugate) is also entire, and hence $N(y) \equiv m(y)m^*(y)$ is also entire with order at most 2 and with a pointwise bound that is independent of f. However, when $y \in \mathbf{R}^n$ (i.e., $v_j = 0$ for all j) then $N(y) = |m(y)|^2$. Conclusion (b) is then an immediate consequence of the relation between f and m which implies that for $y \in \mathbf{R}^n$, $|f(y)|^{2(p-1)} = \|m\|_{p'}^{-2}|m(y)|^2 = \|m\|_{p'}^{-2}N(y)$; thus $|f|^{2(p-1)}$ has an analytic extension of order at most 2, namely $\|m\|_{p'}^{-2}N$. It only has to be shown that $\|m\|_{p'}^{-2}$ is universally bounded, but this follows from the relation $C_{p\to q} = \|\mathscr{G}f\|_q = \int mf = \|m\|_{p'}$.

Conclusion (c) follows from the fact that when $y \in \mathbf{R}^n$ then $|f(y)| = [|m(y)|/\|m\|_{p'}]^{p'-1} \leqq W(y)^{p'-1}\|m\|_{p'}^{1-p'}$. □

The next two lemmas validate the assertion in Sect. I that linear terms can be eliminated from Gaussians.

2.2. Lemma (elimination of linear terms from Gaussian kernels). *Let G be the (degenerate or nondegenerate) Gaussian kernel given in* (1.1) *with positive definite or semidefinite real quadratic form M in* (1.3) *and with real linear term $L \in \mathbf{R}^{2n}$. Let G_0 denote the Gaussian kernel with no linear term, which is obtained from G by setting*

$L=0$, *i.e.*, $G_0(x, y) = G(x, y)\exp\left\{-2\left(L, \begin{pmatrix} x \\ y \end{pmatrix}\right)\right\}$. *Let* $1 \leqq p \leqq \infty$ *and* $1 \leqq q \leqq \infty$. *Then the following conditions are equivalent.*

(*i*) $\mathscr{G}_0$ *is bounded from* $L^p(\mathbf{R}^n)$ *to* $L^q(\mathbf{R}^n)$ *and the equation* $MV = L$ *has a solution* $V \in \mathbf{R}^{2n}$.

(*ii*) $\mathscr{G}$ *is bounded from* $L^p(\mathbf{R}^n)$ *to* $L^q(\mathbf{R}^n)$.

In case these conditions are both satisfied the relation between the norms is

$$C_{p\to q}(G) = C_{p\to q}(G_0)\exp\{(L, V)\}\,.$$

The number (L, V) *is uniquely defined even if the vector* V *is not unique.* $\mathscr{G}$ *has a unique maximizer if and only if* $\mathscr{G}_0$ *has one.*

Proof. (i) $\Rightarrow$ (ii). This was explained in Sect. I. Simply change variables; writing $V = \begin{pmatrix} a \\ b \end{pmatrix}$, let $x \to x + a$ and $y \to y + b$. Then

$$G \to G_0 \exp\left\{(L, V) - 2\mathrm{i}\left(\begin{pmatrix} x \\ y \end{pmatrix}, NV\right) - \mathrm{i}(V, NV)\right\}.$$

The imaginary terms above do not affect the norm. Since M is Hermitian L must be orthogonal to $\mathscr{K} \equiv$ kernel of $M \subset \mathbf{R}^{2n}$, while any two solutions V_1 and V_2 differ by an element of $\mathscr{K}$. Thus, (L, V) is unique. This change of variables also shows that $\mathscr{G}$ has a unique maximizer if and only if $\mathscr{G}_0$ has one.

(ii) $\Rightarrow$ (i). Suppose that $MV = L$ has no solution. Then, since M is Hermitean, L is not orthogonal to $\mathscr{K}$ and thus there is a vector $W = \begin{pmatrix} s \\ t \end{pmatrix} \in \mathscr{K}$ such that the number $P = (W, L)$ is positive. Make the change of variables $x \to x + s$ and $y \to y + t$. Then, since $MW = 0$, G becomes

$$\tilde{G}(x, y) = G(x, y)\exp\left\{-\mathrm{i}(W, NW) - 2\mathrm{i}\left(\begin{pmatrix} x \\ y \end{pmatrix}, NW\right) + 2P\right\}.$$

The change of variables is an isometry so the norm of $\tilde{\mathscr{G}}$ is the same as the norm of $\mathscr{G}$ and, since the imaginary terms are irrelevant, we have $C_{p\to q}(G) = C_{p\to q}(\tilde{G}) = e^{2P}C_{p\to q}(G)$. This is a contradiction since $C_{p\to q}(G) \neq 0$. Thus $MV = L$ has a solution and the same change of variables can be made as before to derive the relation between the norms of $\mathscr{G}$ and $\mathscr{G}_0$. □

2.3. Lemma (elimination of linear terms from maximizers). *Let* G *be a centered Gaussian kernel (degenerate or nondegenerate) and let* $1 < p < \infty$ *and* $1 < q < \infty$. *Assume* $\mathscr{G}: L^p(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$ *is bounded (which is automatically true in the nondegenerate case). If* $g(x) = \exp\{-(x, Jx) + (l, x)\}$ *is a Gaussian function that maximizes* $\mathscr{R}_{p\to q}(g)$ *among all Gaussian functions then* $g_0(x) = \exp\{-(x, Jx)\}$ *is also a maximizer. Moreover, if* $\mathscr{R}_{p\to q}(g)$ *does not have a maximizer among Gaussian functions (which can happen only if* G *is degenerate) then the supremum of* $\mathscr{R}_{p\to q}(g)$ *over Gaussian functions equals the supremum over centered Gaussian functions. Finally, if* G *is nondegenerate then* $g = g_0$, *i.e.*, $l = 0$, *and therefore* g *is centered.*

Proof. Consider the functions $g_\lambda(x) = \exp\{-(x, Jx) + \lambda(l, x)\}$ with λ a real parameter. Clearly $g_\lambda \in L^p(\mathbf{R}^n)$ for all λ and, by a well known property of Gaussian

integrals,

$$\|g_\lambda\|_p = \|g_0\|_p e^{\alpha\lambda^2} \quad \text{and} \quad \|\mathscr{G}g_\lambda\|_q = \|\mathscr{G}g_0\|_q e^{\beta\lambda^2}$$

for some real constants α and β. There are three cases to be considered:

(*i*) $\alpha > \beta$. By setting $\lambda = 0$, $\mathscr{R}_{p\to q}$ is increased, i.e., $\mathscr{R}_{p\to q}(g_0) > \mathscr{R}_{p\to q}(g)$. This means that g is not a maximizer—which is a contradiction.
(*ii*) $\alpha < \beta$. By letting λ tend to infinity we conclude that $\mathscr{R}_{p\to q}$ (and hence also $\mathscr{G}$) is unbounded—which is a contradiction.
(*ii*) $\alpha = \beta$. In this case g_λ is a maximizer for every λ and hence g_0 is a maximizer, as claimed.

These considerations prove all but the last sentence of the lemma.

If G is nondegenerate it is possible to go further. Consider the following sequence of functions with $\lambda = j$, namely $h_j = Z_j g_j$ for $j = 1, 2, 3, \ldots$, where the numbers Z_j are chosen so that $\|h_j\|_p = 1$ for each j. This is a bounded maximizing sequence and, by a trivial modification of the last part of Lemma 2.1 (using the fact that a nonzero $L^p(\mathbf{R}^n)$ weak limit of Gaussian functions is a Gaussian function), there is a nonzero Gaussian function $h \in L^p(\mathbf{R}^n)$ and a subsequence $j(1), j(2), \ldots$ such that $h_{j(k)} \to h$ strongly in $L^p(\mathbf{R}^n)$ as $k \to \infty$. If $l \neq 0$, however, it is easy to check that $h_j \to 0$ weakly in $L^p(\mathbf{R}^n)$ as $j \to \infty$. This contradicts the supposed strong convergence to a nonzero function. □

III. Nondegenerate gaussian kernels

A main ingredient in the following theorems is Minkowski's inequality for integrals. It was exploited by Beckner [B1; B2] to prove that the sharp bound for the tensor product of two operators (e.g., Fourier transforms) is often the product of the individual bounds. In particular, the bound for the Fourier transform from $L^p(\mathbf{R}^n)$ to $L^{p'}(\mathbf{R}^n)$ is $(C_p^B)^n$, where C_p^B is the sharp constant for $\mathbf{R}^1$. A proof of Minkowski's inequality can be found in [HLP]. Of crucial importance here is the sharp form in which the necessary and sufficient condition for equality is specified; this condition was not used before to analyze Gaussian kernels.

3.1. Lemma (Minkowski's inequality). *Let $f: \mathbf{R}^n \times \mathbf{R}^m \to [0, \infty]$ be Lebesgue measurable and let $1 \leqq r < \infty$. Suppose that the measurable function M, defined for almost every $x \in \mathbf{R}^n$ by*

$$M(x) \equiv \int_{\mathbf{R}^m} f(x, y)^r \, dy \,,$$

is finite for almost every x and that $M^{1/r} \in L^1(\mathbf{R}^n)$. Then the measurable function

$$N(y) \equiv \int_{\mathbf{R}^n} f(x, y) \, dx$$

is finite for almost every $y \in \mathbf{R}^m$ and

$$\left\{ \int_{\mathbf{R}^m} N^r \right\}^{1/r} \leqq \int_{\mathbf{R}^n} M^{1/r} \,. \qquad (*)$$

Furthermore, if $r > 1$ and if there is equality in (*) then there are nonnegative, measurable functions $A \in L^1(\mathbf{R}^n)$ and $B \in L^r(\mathbf{R}^m)$ such that

$$f(x, y) = A(x)B(y)$$

for almost every $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$.

Remark. This lemma extends to an arbitrary pair of measure spaces (X, μ) and (Y, ν) in place of $(\mathbf{R}^n, dx)$ and $(\mathbf{R}^m, dy)$ when μ and ν are sigma finite.

As a first application of Minkowski's inequality the uniqueness of maximizers for real, nondegenerate Gaussian kernels for *all* p and q will be proved. This is Case (A) of Section I. It is to be noted that the order of integration in Theorem 3.2 is as in [S] and is opposite to that of Theorem 3.4 and opposite to the order in Beckner's lemma. Analyticity considerations play only a subsidiary role in Theorem 3.2 and can be bypassed if desired, but they are important later. Theorem 3.2 was already essentially contained in [BL] Theorems 7 and 13. The following proof is offered because (*i*) it is different from the [BL] approach and (*ii*) it illustrates the techniques of the present paper.

3.2. Theorem (unique Gaussian maximizer for all p and q in the real nondegenerate case). *Let G be a real, nondegenerate, centered Gaussian kernel, i.e., the matrix N in (1.3) is zero. Let $1 < p < \infty$ and $1 < q < \infty$. Then $\mathscr{G}$ has exactly one maximizer, f, (up to a multiplicative constant) from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ and f is a real, centered Gaussian, i.e., $f(x) = \exp\{-(x, Jx)\}$ with J being a real, positive definite matrix.*

Proof. Consider the linear operator $\mathscr{G}^{(2)} = \mathscr{G} \otimes \mathscr{G} : L^p(\mathbf{R}^{2n}) \to L^q(\mathbf{R}^{2n})$ given by the Gaussian kernel $G^{(2)}((x_1, x_2), (y_1, y_2)) = G(x_1, y_1)G(x_2, y_2)$ with x_1, x_2, y_1 and y_2 in $\mathbf{R}^n$. The first goal is to prove that $C_{p\to q}(G^{(2)}) = C_{p\to q}(G)^2$. If $F \in L^p(\mathbf{R}^{2n})$ then $(y_1, y_2) \mapsto G^{(2)}((x_1, x_2), (y_1, y_2))\, F(y_1, y_2)$ is in $L^1(\mathbf{R}^{2n})$ for every (x_1, x_2) because $G^{(2)}$ is nondegenerate. Fubini's theorem and Minkowski's inequality yield

$$\|\mathscr{G}^{(2)}F\|_q^q = \int \{ \int | \int (\int G(x_1, y_1)G(x_2, y_2)F(y_1, y_2)dy_1)dy_2|^q dx_1 \} dx_2 \tag{1}$$

$$\leqq \int \{ \int [\int G(x_2, y_2)|K(x_1, y_2)|dy_2]^q dx_1 \} dx_2$$

$$\text{(with } K(x_1, y_2) = \int G(x_1, y_1)F(y_1, y_2)dy_1) \tag{2}$$

$$\leqq \int \{ \int [\int G(x_2, y_2)^q |K(x_1, y_2)|^q dx_1]^{1/q} dy_2 \}^q dx_2 \tag{3}$$

$$\leqq (C_{p\to q}(G))^q \int \{ \int G(x_2, y_2) [\int |F(y_1, y_2)|^p dy_1]^{1/p} dy_2 \}^q dx_2 \tag{4}$$

$$\leqq (C_{p\to q}(G))^{2q} \{ \iint |F(y_1, y_2)|^p dy_1\, dy_2 \}^{q/p}. \tag{5}$$

(Notes: (2) → (3) is Minkowski's inequality. (3) → (4) uses $C_{p\to q}(G) \geqq \mathscr{R}_{p\to q}(F(\cdot, y_2))$ for each y_2. (4) → (5) uses $C_{p\to q}(G) \geqq \mathscr{R}_{p\to q}((\int |F(y_1, \cdot)|^p dy_1)^{1/p})$. The fact that $G(x, y) \geqq 0$ is crucial. Here the x_1 integration was done before the x_2 integration; in Theorem 3.4 the x_2 integration will be done first.) Inequalities (1)–(5) establish that $C_{p\to q}(G^{(2)}) \leqq C_{p\to q}(G)^2$. Clearly, by considering F's of the product form $F(y_1, y_2) = h(y_1)h(y_2)$, the reverse inequality is obtained, and so the goal is reached.

Suppose now that $F : \mathbf{R}^{2n} \to \mathbf{C}$ is a maximizer for $G^{(2)}$. Since $G^{(2)}$ is nondegenerate, it has a maximizer by Lemma 2.1. Since $G(x, y) > 0$ for all x and y, it is clear that $F = \lambda|F|$ and $|\lambda| = 1$, for otherwise replacing F by $|F|$ will increase the quotient $\mathscr{R}_{p\to q}$ for $G^{(2)}$. It can be assumed henceforth that $F \geqq 0$. Since F is

a maximizer all the inequalities in (1)–(5) must be equalities. Equality of (2) and (3) implies, by Lemma 3.1, that for almost every x_2 there are measurable functions A_{x_2} and $B_{x_2}: \mathbf{R}^n \to [0, \infty)$ such that

$$G(x_2, y_2)K(x_1, y_2) = A_{x_2}(x_1)B_{x_2}(y_2) \tag{6}$$

for almost every x_1 and y_2. Since $G > 0$, this equation can be divided by $G(x_2, y_2)$ to obtain $K(x_1, y_2) = A_{x_2}(x_1)E_{x_2}(y_2)$ with $E_{x_2}(y) \equiv B_{x_2}(y)/G(x_2, y)$. However, $K(x_1, y_2)$ is independent of x_2 and therefore if any particular value of x_2 is chosen for which (6) holds for almost every x_1 and y_2, and if the functions A and $E: \mathbf{R}^n \to [0, \infty)$ are defined by $A \equiv A_{x_2}$ and $E \equiv E_{x_2}$ for this value of x_2, then

$$K(x_1, y_2) = A(x_1)E(y_2)$$

for almost every x_1 and y_2. If this equation is multiplied by $G(x_2, y_2)$ and integrated over y_2 the result is

$$(\mathscr{G}^{(2)}F)(x_1, x_2) = A(x_1)Z(x_2)$$

for almost every x_1 and x_2 with $Z = \mathscr{G}E$. Since $G^{(2)} > 0$, both A and Z are strictly positive functions.

There is a function $H \in L^{q'}(\mathbf{R}^{2n})$ with $\|H\|_{q'} = 1$, such that $\|\mathscr{G}^{(2)}F\|_q = \int H \cdot \mathscr{G}^{(2)}F$. In fact

$$H(x_1, x_2) = (\text{const.})[(\mathscr{G}^{(2)}F)(x_1, x_2)]^{q-1} = (\text{const.})A(x_1)^{q-1}Z(x_2)^{q-1}.$$

The point here is that H is a product function. Then, as in the proof of Lemma 2.1, F satisfies

$$F(y_1, y_2) = (\text{const.})\left\{\iint G(x_1, y_1)G(x_2, y_2)H(x_1, x_2)\,dx_1\,dx_2\right\}^{p'-1} = \alpha(y_1)\beta(y_2) \tag{7}$$

for some positive function α and $\beta: \mathbf{R}^n \to [0, \infty)$. In brief, F *must be* a product function, and this fact is crucial for the next step.

One example of a maximizer is $F(y_1, y_2) = f(y_1)f(y_2)$, where f is an $L^p(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$ maximizer for G (whose existence is guaranteed by Lemma 2.1). For the reason given before about F, we can and do assume that $f(x) \geqq 0$ for all $x \in \mathbf{R}^n$.

A more interesting maximizer is

$$F(y_1, y_2) = f\left(\frac{y_1 - y_2}{\sqrt{2}}\right)f\left(\frac{y_1 + y_2}{\sqrt{2}}\right). \tag{8}$$

Here, the essential property of $O(2)$ rotation invariance of products of centered Gaussians and of Lebesgue measure is being exploited. If θ is any fixed angle and if x_1', x_2', y_1', y_2' in $\mathbf{R}^n$ are defined by $x_1' = x_1 \cos\theta - x_1 \sin\theta$, $x_2' = x_1 \sin\theta + x_2 \cos\theta$, $y_1' = y_1 \cos\theta - y_2 \sin\theta$, $y_2' = y_1 \sin\theta + y_2 \cos\theta$, the $O(2)$ invariance of Lebesgue measure is that $dx_1\,dx_2 = dx_1'\,dx_2'$ and $dy_1\,dy_2 = dy_1'\,dy_2'$. The $O(2)$ invariance of centered Gaussian functions is that $g(x_1)g(x_2) = g(x_1')g(x_2')$, while for centered Gaussian kernels $G(x_1, y_1)G(x_2, y_2) = G(x_1', y_1')G(x_2', y_2')$. With the choice $\theta = \pi/4$, these observations lead to (8). Combining (7) and (8),

$$f\left(\frac{y_1 - y_2}{\sqrt{2}}\right)f\left(\frac{y_1 + y_2}{\sqrt{2}}\right) = \alpha(y_1)\beta(y_2). \tag{9}$$

for almost every y_1 and y_2.

Equation (9) implies that f is a Gaussian. Instead of proving this in full generality for $L^p(\mathbf{R}^n)$ functions, as is done by Carlen [CA], it is easier to simplify the proof here by taking the $2(p-1)^{th}$ power of (9) and by taking advantage of the analyticity result Lemma 2.1(b). Introducing $h = f^{2(p-1)}$, $\gamma = \alpha^{2(p-1)}$ and $\delta = \beta^{2(p-1)}$, it is seen from (9) (by fixing y_2) that γ is analytic; likewise δ is analytic. Thus, (9) holds for *all* y_1 and y_2 because when two analytic functions on $\mathbf{C}^n \times \mathbf{C}^n$ agree almost everywhere on $\mathbf{R}^n \times \mathbf{R}^n$ then they agree everywhere. Furthermore f never vanishes for real y because if $f(Y) = 0$ then, setting $y_1 = y_2 + \sqrt{2}\,Y$, we would have that $0 = \gamma(y_2 + \sqrt{2}\,Y)\delta(y_2)$ for all y_2; this is impossible, given that γ and δ are analytic, unless $\gamma \equiv 0$ or $\delta \equiv 0$, which contradicts the assumption that $f \not\equiv 0$. Thus, the logarithms of h, γ and δ are real analytic and

$$\ln\left[h\left(\frac{y_1 - y_2}{\sqrt{2}}\right)\right] + \ln\left[h\left(\frac{y_1 + y_2}{\sqrt{2}}\right)\right] = \ln[\gamma(y_1)] + \ln[\delta(y_2)]\,. \tag{10}$$

If ∂_i denotes the derivative with respect to the i^{th} coordinate, and ∂_i with respect to y_1 and ∂_j with respect to y_2 is taken in (10), then

$$(\partial_i \partial_j \ln h)\left(\frac{y_1 - y_2}{\sqrt{2}}\right) = (\partial_i \partial_j \ln h)\left(\frac{y_1 + y_2}{\sqrt{2}}\right),$$

which implies that the function $\partial_i \partial_j \ln h$ is a constant (call it $4(1-p)J_{ij}$) and therefore $\ln[f(y)] = \dfrac{1}{2(p-1)} \ln[h(y)] = -(y, Jy) + (l, y)$ for some vector l. According to Lemma 2.3, $l = 0$ since G is centered and nondegenerate. This completes the proof that f must be a centered Gaussian.

It remains to prove that f is unique (i.e., the matrix J above is unique). One way would be to compute $\mathscr{R}_{p\to q}(\exp\{-(x, Jx)\})$ for G and then deduce that there is only one optimum J. A very much easier route is to suppose that there are two maximizers f^1 and f^2 with $f^i(y) = \exp\{-(y, J^i y)\}$. Then, for the same reason as before ($O(2)$ symmetry) the function

$$F(y_1, y_2) = f^1\left(\frac{y_1 - y_2}{\sqrt{2}}\right) f^2\left(\frac{y_1 + y_2}{\sqrt{2}}\right) \tag{11}$$

is a maximizer for $\mathscr{G}^{(2)}$. There are two ways in which this implies that $f^1 = f^2$. The first is to use (7), namely F must be a product function, and to note that this product structure is true if and only if $J^1 = J^2$. The second way is to note that since the F in (11) is never zero and, since (3) $\to$ (4) must be an equality, we have that the function $y_1 \mapsto h_{y_2}(y_1) \equiv F(y_1, y_2)$ must be a maximizer for $\mathscr{G}$ for almost every y_2. Although the function h_{y_2} is a Gaussian for each y_2, the Gaussian will have a linear term for each $y_2 \neq 0$ unless $J^1 = J^2$. However, Lemma 2.3 precludes the existence of such a linear term, so $J^1 = J^2$. □

The next theorem concerns Case (B) of Sect. I.

3.3. Theorem (unique Gaussian maximizers in the imaginary, nondegenerate case). *Let G be a centered, nondegenerate Gaussian kernel with a real diagonal part and a purely imaginary off-diagonal part, i.e.,*

$$G(x, y) = \exp\{-(x, Ax) - (y, By) - 2i(x, Dy)\}$$

where A, B and D are real n × n matrices and A and B are positive definite. Let $1 < p \leqq 2$ *and* $1 < q < \infty$ *or else* $1 < p < \infty$ *and* $2 \leqq q \leqq \infty$. *Then, in either case,* $\mathscr{G}$ *has exactly one maximizer, f, (up to a multiplicative constant) from* $L^p(\mathbf{R}^n)$ *to* $L^q(\mathbf{R}^n)$ *and this f is a real, centered Gaussian, i.e.,* $f(x) = \exp\{-(x, Jx)\}$ *with J being a real, positive definite matrix.*

Proof. Assume at first that D is nonsingular. Since A and B are positive definite there are nonsingular real matrices U and V so that the change of variables $x \to Ux$ and $y \to Vy$ changes A and B to the identity matrix, I, that is $I = U^T A U = V^T B V$, where T denotes transpose. Then $(x, Dy) \to (x, \tilde{D}y)$ with $\tilde{D} = U^T D V$. The polar decomposition of $\tilde{D}$ is $\tilde{D} = W|\tilde{D}|$, where W is orthogonal and $|\tilde{D}|$ is positive definite (the assumption that D is nonsingular is used here). Then there is an orthogonal matrix Y such that $Y^T|\tilde{D}|Y$ is diagonal and there is a real diagonal matrix Z such that $ZY^T|\tilde{D}|YZ = I$. Now make one more change of variables: $x \to WYZx$ and $y \to YZy$ so that $(x, \tilde{D}y) \to (WYZx,\ W|\tilde{D}|YZy) = (x, y)$ and $(x, x) = (x, Ix) \to (WYZx,\ WYZx) = (x, Z^2x)$ and $(y, y) \to (YZy, YZy) = (y, Z^2y)$. These two changes of variables affect $\mathscr{R}_{p\to q}$ in a trivial way (involving only p and q and the determinants of U, V and Z) and, most importantly, take Gaussian functions into Gaussian functions. In short, it can be assumed without loss of generality that G has the canonical form

$$G(x, y) = \exp\{-(x, Ax) - (y, Ay) - 2i(x, y)\}\,, \tag{1}$$

where A is positive definite and diagonal.

By duality $C_{p\to q}(G) = C_{q'\to p'}(G^T)$ with $G^T(x,\ y) \equiv G(y, x) = G(x, y)$, so it suffices to consider only the case $1 < p \leqq 2$ and $1 < q < \infty$. It is easily seen that $(\mathscr{G}f)(x) = \exp\{-(x, Ax)\}\hat{h}(x)$ where $\hat{h}$ is the Fourier transform of the function $h(y) = \exp\{-(y, Ay)\}f(y)$. Since $f \in L^p(\mathbf{R}^n)$ it has a Fourier transform $\hat{f}$, and Beckner's theorem (which will also be proved here in Theorem 4.1 and 4.2(1)) states that $\|\hat{f}\|_{p'} \leqq (C_p^B)^n \|f\|_p$, where Beckner's constant C_p^B is the sharp constant for the $p \to p'$ norm of the Fourier transform in $\mathbf{R}^1$. By the convolution formula, $\hat{h}$ satisfies

$$\hat{h}(x) = \mu \int \exp\{-(x - y, A^{-1}(x - y))\}\hat{f}(y)\mathrm{d}y\,,$$

where $\mu > 0$ is a constant which depends only on A. Therefore $(\mathscr{G}f)(x) = \mu(\tilde{\mathscr{G}}\hat{f})(x)$ where $\tilde{G}$ is the real, centered, nondegenerate Gaussian

$$\tilde{G}(x, y) = \exp\{-(x, Ax) - (x - y, A^{-1}(x - y))\}\,. \tag{2}$$

Thus

$$\mathscr{R}_{p\to q}(G, f)\|f\|_p = \mu\mathscr{R}_{p'\to q}(\tilde{G}, \hat{f})\|\hat{f}\|_{p'} \leqq \mu C_{p'\to q}(\tilde{G})\|\hat{f}\|_{p'} \leqq \mu C_{p'\to q}(\tilde{G})(C_p^B)^n\|f\|_p\,, \tag{3}$$

from which it follows that $C_{p\to q}(G) \leqq \mu(C_p^B)^n C_{p'\to q}(\tilde{G})$. However, equality can be achieved in (3) in exactly one way (up to a multiplicative constant). By Theorem 3.2 there is exactly one choice for $\hat{f}$ that will make the first inequality in (3) into an equality. This $\hat{f}$ is a real, centered Gaussian, $\hat{f}(x) = \exp\{-(x, Jx)\}$. Its inverse transform f is also a real, centered Gaussian, i.e., $f(x) = (\text{const.})\exp\{-(x, J^{-1}x)\}$. The second inequality in (3) (Beckner's) is an equality for any *real* Gaussian (in particular, our f), and therefore f is the unique maximizer as asserted in the theorem.

In case D is singular, a change of variables similar to the above replaces the canonical form (1) by

$$G(x, y) = \exp\{-(x, Ax) - (y, Ay) - 2i(x, Py)\}$$

where $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ is a diagonal projection onto $\mathbf{R}^m$ (with $m < n$ being the rank of D) and $A = \begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix}$ with α positive definite, $m \times m$ and diagonal. Writing $x \in \mathbf{R}^n$ as (x_1, x_2) with $x_1 \in \mathbf{R}^m$ and $x_2 \in \mathbf{R}^{n-m}$, define $g: \mathbf{R}^m \to \mathbf{C}$ by

$$g(y_1) = \int_{\mathbf{R}^{n-m}} \exp\{-(y_2, y_2)\} f(y_1, y_2) \mathrm{d}y_2 \,, \tag{4}$$

and $G^*: \mathbf{R}^m \times \mathbf{R}^m \to (0, \infty)$ by $G^*(x_1, y_1) = \exp\{-(x_1, \alpha x_1) - (y_1, \alpha y_1) - 2i(x_1, y_1)\}$. Then, using Fubini's theorem and the same analysis as before with α and m in place of A and n, and with $\tilde{G}^*: \mathbf{R}^m \times \mathbf{R}^m \to (0, \infty)$ as given in (2),

$$\mathscr{R}_{p\to q}(G, f) \| f \|_{L^p(\mathbf{R}^n)} = \nu \mathscr{R}_{p\to q}(G^*, g) \| g \|_{L^p(\mathbf{R}^m)} \leqq \nu \mu C_{p'\to q}(\tilde{G}^*)(C_p^B)^m \| g \|_{L^p(\mathbf{R}^m)} \tag{5}$$

where ν is the $L^q(\mathbf{R}^{n-m})$ norm of the Gaussian function $\exp\{-(x_2, x_2)\}$. As was the case for (3) and the subsequent argument, equality in (5) is uniquely achieved by a real, centered Gaussian

$$g(y_1) = \exp\{-(y_1, Ey_1)\} \,, \tag{6}$$

where E is a real, positive definite $m \times m$ matrix. By Hölder's or Minkowski's inequality, it follows from (4) that

$$\| g \|_{L^p(\mathbf{R}^m)} \leqq N \| f \|_{L^p(\mathbf{R}^n)} \tag{7}$$

where N is the $L^{p'}(\mathbf{R}^{n-m})$ norm of the Gaussian function $\exp\{-(y_2, y_2)\}$. Equality in (7) is compatible with (6) if and only if $f(y) = f(y_1, y_2) = (\text{const.}) \exp\{-y_1, Ey_1) - (p'-1)(y_2, y_2)\}$. The reason is that equality in (7) requires that $f(y_1, y_2) = \lambda(y_1) \exp\{-(p'-1)(y_2, y_2)\}$ for almost every $y_1 \in \mathbf{R}^m$. Then, computing the integral in (4), one finds that $\exp\{-(y_1, Ey_1)\} = g(y_1) = (\text{const.})\, \lambda(y_1)$. □

Finally, Case (C) of Sect. I will be considered.

3.4. Theorem (unique Gaussian maximizer when $p \leqq q$ in the general nondegenerate case). *Let G be a centered, nondegenerate Gaussian kernel and let $1 < p \leqq q < \infty$. Then $\mathscr{G}$ has exactly one maximizer, f, (up to a multiplicative constant) from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ and f is a centered Gaussian function.*

Proof. As in the proof of Theorem 3.2, the key is to study the kernel $G^{(2)} = G \otimes G$ by means of Minkowski's inequality. Now, however, the x_2 integration is done first. Thus, for $F \in L^p(\mathbf{R}^{2n})$,

$$\begin{aligned}
\| \mathscr{G}^{(2)} F \|_q^q &= \int \{ \int | \int G(x_2, y_2) (\int G(x_1, y_1) F(y_1, y_2) \mathrm{d}y_1) \mathrm{d}y_2 |^q \mathrm{d}x_2 \} \mathrm{d}x_1 && (1) \\
&= \int \{ \int | \int G(x_2, y_2) K(x_1, y_2) \mathrm{d}y_2 |^q \mathrm{d}x_2 \} \mathrm{d}x_1 \\
&\quad (\text{with } K(x_1, y_2) = \int G(x_1, y_1) F(y_1, y_2) \mathrm{d}y_1) && (2) \\
&\leqq (C_{p\to q}(G))^q \int \{ \int |K(x_1, y_2)|^p \mathrm{d}y_2 \}^{q/p} \mathrm{d}x_1 && (3) \\
&\leqq (C_{p\to q}(G))^q \{ \int [\int |K(x_1, y_2)|^q \mathrm{d}x_1]^{p/q} \mathrm{d}y_2 \}^{q/p} && (4) \\
&\leqq (C_{p\to q}(G))^{2q} \{ \iint |F(y_1, y_2)|^p \mathrm{d}y_1 \mathrm{d}y_2 \}^{q/p} \,. && (5)
\end{aligned}$$

(Notes: (2) → (3) uses $C_{p\to q}(G) \geqq \mathscr{R}_{p\to q}(K(x_1,\cdot))$ for each x_1. (3) → (4) is Minkowski's inequality for the exponent $r = q/p \geqq 1$ and the function $|K(x_1, y_2)|^p$. (4) → (5) uses $C_{p\to q}(G) \geqq \mathscr{R}_{p\to q}(F(\cdot, y_2))$ for each y_2.) This inequality (along with consideration of F's of the form $F(y_1, y_2) = h(y_1)h(y_2)$) shows that

$$C_{p\to q}(G^{(2)}) = C_{p\to q}(G)^2 .$$

Suppose now that $F: \mathbf{R}^{2n} \to \mathbf{C}$ is a maximizer for $\mathscr{G}^{(2)}$. Since $G^{(2)}$ and G are nondegenerate, maximizers exist for each of them by Lemma 2.1. Then all the inequalities in (1)–(5) must be equalities. In particular, inequality (4) → (5) implies that the function $y_1 \mapsto F(y_1, y_2)$ must either be the zero function or it must be a maximizer for $\mathscr{G}$ for almost every $y_2 \in \mathbf{R}^n$. (It is well known that this function is in $L^p(\mathbf{R}^n)$ for almost every y_2.) As in the proof of Theorem 3.2, the $O(2)$ invariance of $G^{(2)}$ implies that the function given by

$$F(y_1, y_2) = f\left(\frac{y_1 - y_2}{\sqrt{2}}\right) f\left(\frac{y_1 + y_2}{\sqrt{2}}\right) \tag{6}$$

is a maximizer for $\mathscr{G}^{(2)}$ when f is a maximizer for $\mathscr{G}$, as will henceforth be assumed. Thus, for almost every z in $\mathbf{R}^n$, the function $g_z(y) = F(y, z)$ is in $L^p(\mathbf{R}^n)$ and either (a) it is a maximizer for $\mathscr{G}$ or (b) g_z is the zero function. The second possibility (b) can be excluded by Lemma 2.1 (b). If $g_z \equiv 0$ then, from (6), $f(w) = 0$ for all w in some set $A \subset \mathbf{R}^n$ of positive Lebesgue measure. But $|f|^{2(p-1)}$ is analytic and this is impossible unless $f \equiv 0$. Thus it can be assumed that g_z is indeed a maximizer for almost every z, i.e., $g_z \neq 0$.

In fact g_z is an $L^p(\mathbf{R}^n)$ maximizer for *every* $z \in \mathbf{R}^n$. To prove this assertion, fix z and let $z_1, z_2, \ldots$ be any sequence in $\mathbf{R}^n$ such that $z_j \to z$ as $j \to \infty$ and such that g_{z_j} is an $L^p(\mathbf{R}^n)$ maximizer for each j. Such a sequence exists because g_z is a maximizer for z's in a dense set. Define $h_j(y) = Z_j g_{z_j}(y)$ where Z_j is chosen so that $\|h_j\|_p = 1$ for each j. By Lemma 2.1, there is a subsequence (still denoted by h_j) and a maximizing function $h \in L^p(\mathbf{R}^n)$ such that $h_j \to h$ strongly as $j \to \infty$. By passing to a further subsequence this convergence can also be assumed to be pointwise almost everywhere. However, translation is a continuous operation in $L^p(\mathbf{R}^n)$ and thus, by passing to a further subsequence, $f((y + z_j)/\sqrt{2})$ converges pointwise to $f((y + z)/\sqrt{2})$ for almost every y. Likewise, by passing to a further subsequence, $f((y - z_j)/\sqrt{2})$ converges pointwise to $f((y - z)/\sqrt{2})$ for almost every y. It follows then that the maximizer h satisfies

$$h(y) = f\left(\frac{y - z}{\sqrt{2}}\right) f\left(\frac{y + z}{\sqrt{2}}\right) \lim_{j\to\infty} Z_j$$

for almost every y. Therefore $\lim_{j\to\infty} Z_j$ exists and g_z is a maximizer for every $z \in \mathbf{R}^n$.

Our first application of this result will be the proof that there is a Gaussian maximizer. Take $z = 0$ so that $f^{(2)}(y) \equiv f(y/\sqrt{2})^2$ is a maximizer. Then apply the same conclusion to $f^{(2)}$ so that $f^{(4)}(y) \equiv f(y/2)^4$ is also a maximizer. Repeating this indefinitely, the sequence of $L^p(\mathbf{R}^n)$ functions given by

$$g_j(y) = N_j f\left(\frac{y}{\sqrt{j}}\right)^j \tag{7}$$

is a sequence of maximizers for $j = 2, 4, 8, 16, \ldots$. The number N_j is chosen in each case so that $\|g_j\|_p = 1$. Using Lemma 2.1 again we infer the existence of a subsequence (still denoted by j) and a maximizer g such that $g_j \to g$ strongly in $L^p(\mathbf{R}^n)$ and pointwise almost everywhere. Our goal will be to prove that g is a Gaussian. This can be inferred from the central limit theorem, but the following argument is more direct and will be needed later for the proof that *every* maximizer is a Gaussian.

The first step is to prove that $f(0) \neq 0$. Recall from Lemma 2.1(b) that $R \equiv |f|^{2(p-1)}$ is analytic. Likewise $S \equiv |g|^{2(p-1)}$ is also analytic and

$$S(y) = \lim_{j\to\infty} N_j^{2(p-1)} R\left(\frac{y}{\sqrt{j}}\right)^j \tag{8}$$

for almost all $y \in \mathbf{R}^n$. Since $S_j(y) \equiv N_j^{2(p-1)} R(y/\sqrt{j})^j$ is the $2(p-1)^{th}$ power of the modulus of a maximizer with unit $L^p(\mathbf{R}^n)$ norm (namely g_j), Lemma 2.1(b) states that the analytic extension of S_j is uniformly bounded on compact subsets of $\mathbf{C}^n$. The almost everywhere convergence in (8) then implies (by Vitali's theorem) that (8) holds for all $y \in \mathbf{C}^n$ and that all partial derivatives with respect to y of the sequence of functions S_j also converge as $j \to \infty$ to the corresponding derivatives of S. However, it is easily seen by Leibniz's rule that if $R(0) = 0$ then *every* derivative of S_j at $y = 0$ converges to zero as $j \to \infty$. This is impossible unless $S(y)$ vanishes identically, which contradicts the fact that $\|g\|_p = 1$.

The second step is to prove that g is a Gaussian. By Lemma 2.1(a), for $y \in \mathbf{R}^n$, $f(y) = |m(y)|^{p'}/m(y)$, where $m: \mathbf{C}^n \to \mathbf{C}$ is entire analytic. Since $f(0) \neq 0$, also $m(0) \neq 0$ and hence there is a neighborhood U of $0 \in \mathbf{C}^n$ on which f has an analytic extension and on which f is never zero. [Reason: $m_1(y) \equiv \mathrm{Re}(m(y))$ can be written as a Taylor series for $y \in \mathbf{R}^n$, and so can $m_2(y) \equiv \mathrm{Im}(m(y))$. Consequently m_1 and m_2 extend to entire functions. Then $(m_1^2 + m_2^2)^{p'/2}$ is analytic on U since $m_1(0)^2 + m_2(0)^2 = |m(0)|^2 \neq 0$.] Therefore f has a logarithm, H, which is analytic on U, i.e., $f(y) = f(0)\exp\{H(y)\}$. The function H can be written as $H(y) = (V, y) - (y, Jy) + O(y^3)$ for some $V \in \mathbf{C}^n$ and J a symmetric matrix. For each $y \in \mathbf{R}^n$, the point $y/\sqrt{j}$ lies in U for all sufficiently large j and therefore, by (7),

$$g(y) = \lim_{j\to\infty} N_j f(0)^j \exp\{\sqrt{j}(V, y) - (y, Jy) + O(y^3 j^{-1/2})\}$$

for almost every $y \in \mathbf{R}^n$. The factor $\exp\{O(y^3 j^{-1/2})\}$ converges to 1 as $j \to \infty$ and thus

$$g(y) = \exp\{-(y, Jy)\} \lim_{j\to\infty} N_j f(0)^j \exp\{\sqrt{j}(V, y)\}\,.$$

Clearly this last limit can exist for almost every y if and only if $V = 0$ and $N_j f(0)^j$ has a finite limit (which cannot be zero since $\|g\|_p = 1$). This proves that g must be a Gaussian as claimed (and hence $\mathrm{Re}(J)$ is positive definite) but we also note that the argument also proves the following three statements: Whenever f is a maximizer then (*i*) f is analytic in some complex neighborhood of 0; (*ii*) $f(0) \neq 0$; (*iii*) $(\partial f/\partial y^i)(0) = 0$, for $i = 1, \ldots, n$.

The second assertion of the theorem is that every other maximizer, f, is proportional to the one just found, namely $g(y) = \exp\{-(y, Jy)\}$. Instead of (6) take

$$F(y_1, y_2) = g\left(\frac{y_1 - y_2}{\sqrt{2}}\right) f\left(\frac{y_1 + y_2}{\sqrt{2}}\right)$$

which is obviously also a maximizer for $\mathscr{G}^{(2)}$. By the same reasoning as before, F has the property that $y \mapsto k_z(y) \equiv F(y, z)$ is a maximizer for each fixed $z \in \mathbf{R}^n$. By the three statements just made above, we conclude that k_z is analytic near 0,

$$k_z(0) \neq 0 \quad \text{and} \quad (\partial k_z/\partial y^i)(0) = 0 .$$

This is equivalent to the statement that for every $z \in \mathbf{R}^n$, f is analytic near $z/\sqrt{2}$, $f(z/\sqrt{2}) \neq 0$ and

$$(\partial f/\partial y^i)\left(\frac{z}{\sqrt{2}}\right) = [-Jz]_i f\left(\frac{z}{\sqrt{2}}\right),$$

which shows that $f = g$. □

IV. Degenerate Gaussian kernels

In the three cases (A), (B) and (C) of Sect. I, which correspond to Theorems 3.2, 3.3 and 3.4, every nondegenerate Gaussian kernel has a unique maximizer which is a Gaussian function. By taking suitable limits the following formula 4.1(*), which is one of the main results of this paper, can be deduced for the $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ norm of degenerate kernels. This formula is, of course, trivially true in the nondegenerate case.

4.1. Theorem (the sharp bound for degenerate kernels). *Let G be a centered Gaussian kernel as in* (1.1) *with $L = 0$ and let p and q satisfy the appropriate conditions given in* (A), (B) *or* (C) *of Sect. I, according to the properties of G. Then $\mathscr{G}$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ if and only if the following supremum is finite, in which case the supremum is equal to $C_{p\to q}$.*

$$\sup_g \mathscr{R}_{p\to q}(g) = C_{p\to q}, \tag{*}$$

where the supremum is taken over all centered Gaussian functions, and in Cases (A) *and* (B) *they can be restricted to be real.*

Proof. For each $\varepsilon > 0$ let $h_\varepsilon(x) \equiv \exp\{-\varepsilon(x, x)\}$ and define $G_\varepsilon(x, y) \equiv G(x, y)h_\varepsilon(x)h_\varepsilon(y)$, which is nondegenerate. Correspondingly, there is the linear operator $\mathscr{G}_\varepsilon$. For each $f \in L^p(\mathbf{R}^n)$

$$\begin{aligned}\mathscr{R}_{p\to q}(G_\varepsilon, f)\|f\|_p = \|h_\varepsilon \mathscr{G}(h_\varepsilon f_\varepsilon)\|_q &< \|\mathscr{G}(h_\varepsilon f_\varepsilon)\|_q \leqq C_{p\to q}(G)\|h_\varepsilon f_\varepsilon\|_p \\ &< C_{p\to q}(G)\|f\|_p . \end{aligned} \tag{1}$$

This proves that $C_{p\to q}(G_\varepsilon) < C_{p\to q}(G)$.

On the other hand, assuming that $C_{p\to q}(G) < \infty$, for each $\delta > 0$ there is an $f^\delta \in L^p(\mathbf{R}^n)$ with $\|f^\delta\|_p = 1$ such that $\|\mathscr{G} f^\delta\|_q > C_{p\to q}(G) - \delta$. Then

$$C_{p\to q}(G_\varepsilon) \geqq \mathscr{R}_{p\to q}(G_\varepsilon, f^\delta) = \|\mathscr{G}_\varepsilon(f^\delta)\|_q = \|h_\varepsilon \mathscr{G}(h_\varepsilon f^\delta)\|_q . \tag{2}$$

As $\varepsilon \to 0$, $h_\varepsilon f^\delta \to f^\delta$ strongly in $L^p(\mathbf{R}^n)$, so $\mathscr{G}(h_\varepsilon f^\delta) \to \mathscr{G}(f^\delta)$ strongly in $L^q(\mathbf{R}^n)$. This implies that $h_\varepsilon \mathscr{G}(h_\varepsilon f^\delta) \to \mathscr{G}(f^\delta)$ strongly in $L^q(\mathbf{R}^n)$ as well, and thus, from (2), $\liminf_{\varepsilon\to 0} C_{p\to q}(G_\varepsilon) \geqq C_{p\to q}(G) - \delta$. Since δ is arbitrary, and in view of (1),

$$\lim_{\varepsilon\to 0} C_{p\to q}(G_\varepsilon) = C_{p\to q}(G) . \tag{3}$$

A similar argument shows that (3) holds even if $C_{p\to q}(G) = \infty$.

Now let g_ε denote the maximizer for G_ε, which is a centered Gaussian function. Assume $\|g_\varepsilon\|_p = 1$. Then

$$\|h_\varepsilon g_\varepsilon\|_p C_{p\to q}(G) \geqq \|h_\varepsilon g_\varepsilon\|_p \mathscr{R}_{p\to q}(G, h_\varepsilon g_\varepsilon) = \|\mathscr{G}(h_\varepsilon g_\varepsilon)\|_q$$
$$\geqq \|h_\varepsilon \mathscr{G}(h_\varepsilon g_\varepsilon)\|_q = \|\mathscr{G}_\varepsilon(g_\varepsilon)\|_q = C_{p\to q}(G_\varepsilon)\,. \tag{4}$$

Assuming $\mathscr{G}$ to be bounded, (4) together with (3) and the fact that $\|h_\varepsilon g_\varepsilon\|_p \leqq \|g_\varepsilon\|_p = 1$ implies that $\|h_\varepsilon g_\varepsilon\|_p \to 1$ as $\varepsilon \to 0$. Then

$$C_{p\to q}(G) = \lim_{\varepsilon\to 0} C_{p\to q}(G_\varepsilon) \leqq \lim_{\varepsilon\to 0} \mathscr{R}_{p\to q}(G, h_\varepsilon g_\varepsilon) \leqq C_{p\to q}(G)\,. \tag{5}$$

This proves the theorem in the bounded case since $h_\varepsilon g_\varepsilon$ is a Gaussian function (which is real in Cases (A) and (B)).

In case $\mathscr{G}$ is unbounded, (4) and (5) imply that

$$\infty = \lim_{\varepsilon\to 0} C_{p\to q}(G_\varepsilon) \leqq \lim_{\varepsilon\to 0} \|h_\varepsilon g_\varepsilon\|_p \mathscr{R}_{p\to q}(G, h_\varepsilon g_\varepsilon) \leqq \lim_{\varepsilon\to 0} \mathscr{R}_{p\to q}(G, h_\varepsilon g_\varepsilon)$$

which proves the theorem since $h_\varepsilon g_\varepsilon$ is a Gaussian function. □

4.2. Remarks and examples. Theorem 4.1(∗) is a formula for the $L^p(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$ norm of $\mathscr{G}$. The same formula is, of course, also valid for nondegenerate kernels, but in that case we are assured that there is precisely one g that achieves the supremum. In the degenerate case a maximizer may not exist—*even if $\mathscr{G}$ is bounded*—as the examples below show. In any event, the evaluation of this formula is, in general, a difficult nonlinear algebraic exercise, although it is simple in many applications.

For example, when $G(x, y) = \exp\{2i(x, y)\}$ (the Fourier transform kernel), it is easy to deduce from 4.1(∗) that G is bounded if and only if $q = p' \geqq 2$, in which case a Gaussian function is a maximizer if and only if it has the form

$$g(x) = \mu \exp\{-(x, Jx) + (l, x)\}$$

with J positive definite, real and symmetric and $l \in \mathbf{C}$. Both J and l are arbitrary. This g is not necessarily centered even though G is. In the degenerate case it is not asserted that every maximizer must be centered when G is centered. The sharp constant is then $C_{p\to p'} = (C_p^B)^n$ with

$$C_p^B = \pi^{1/p'} p^{1/2p} (p')^{-1/2p'}\,. \tag{1}$$

[Note: The Fourier transform is an example of *both* Cases (B) and (C). While the proof of Theorem 3.3 (Case (B)) required 4.1(∗) and 4.2(1), the proof of Theorem 3.4 (Case (C)) did not. Therefore no circular reasoning is involved because 3.4 ⇒ 4.1(∗) for Case (C) ⇒ 4.2(1) ⇒ 3.3 ⇒ 4.1(∗) for Case (B).]

Another example is the (real convolution operator $G(x, y) = \exp\{-\lambda(x - y, x - y)\}$ which, using Theorem 4.1, turns out to be bounded if and only if $p \leqq q$ (see [BL] Section 4 for more details). There is a maximizing Gaussian function if and only if $p < q$ and it must have the form

$$g(x) = \exp\{-J(x, x) + (l, x)\} \tag{2}$$

with $J = \lambda\left(\frac{p'}{q'} - 1\right)$ and with $l \in \mathbf{R}^n$ arbitrary. Also

$$(C_{p\to q})^{2/n} = \left(\frac{\pi}{\lambda}\right)^{1/p' + 1/q} (p' - q')^{1/p - 1/q} p^{1/p} (q')^{1/p'} q^{-1/q} (p')^{-1/q'} . \tag{3}$$

When $p = q$ the limiting value $C_{p\to q} = \pi/\lambda$ is correct but, since $J = 0$ in this case, there is no Gaussian maximizer. Indeed, there is *no maximizer of any kind* in this case. To prove this, note that $G(x, y) = H(x - y)$ with $H(x) = \exp\{-\lambda(x, x)\}$ and $\int H(x - y) f(y)\mathrm{d}y = \int f(x - y) H(y)\mathrm{d}y$. Then, by Minkowski's inequality,

$$\begin{aligned}\{\int |\int f(x - y) H(y)\mathrm{d}y|^p \mathrm{d}x\}^{1/p} &\leqq \int \{\int |f(x - y)|^p H(y)^p \mathrm{d}x\}^{1/p} \mathrm{d}y \\ &= \|f\|_p \int H(y)\mathrm{d}y = \left(\frac{\pi}{\lambda}\right)^{n/2} \|f\|_p . \end{aligned} \tag{4}$$

Since the condition in Lemma 3.1 for equality is clearly not satisfied, and since $(\pi/\lambda)^{n/2}$ has already been shown to be the sharp bound, a maximizer cannot exist.

A second example of a degenerate G that is bounded but does not have a maximizer is the following modification of the Fourier transform in $\mathbf{R}^1$ with $\lambda > 0$.

$$G_\lambda(x, y) = \exp\{-\lambda y^2 - 2\mathrm{i}xy\} . \tag{5}$$

It is easily verified for all p that $\mathscr{R}_{p\to q}(g)$ is unbounded on *complex* Gaussian functions when $q < 2$. Thus, it can be assumed that $q \geqq 2$, which places us in Case (B) of Sect. I. If $f_J(x) = \exp\{-Jx^2\}$ is an arbitrary Gaussian function, one finds that when $q \geqq 2$ the optimum choice is J real and

$$[\mathscr{R}_{p\to q}(f_J)]^2 = \pi^{1/q + 1/p'} p^{1/p} q^{-1/q} J^{1/p} (\lambda + J)^{-1/q'} . \tag{6}$$

By maximizing this with respect to J one finds that $C_{p\to q}$ is finite whenever $p \geqq q'$ and $C_{p\to q} = \infty$ when $p < q'$. If $p = q'$ there is no J that maximizes the right side of (6) (i.e., $J \to \infty$), although the right side is bounded. Indeed, *there is no maximizer* of any kind when $p = q'$. If there were a maximizer $f \in L^p(\mathbf{R}^1)$ then, by imitating the proof of Theorem 4.1, it is easily seen that $C_{p\to p'}(G_{\mu\lambda}) > C_{p\to p'}(G_\lambda)$ when $0 < \mu < 1$. This contradicts the conclusion of Theorem 4.1 which states that the supremum over J of the right side of (6) correctly gives $C_{p\to p'}(G_\lambda)$ for every λ, but this supremum is obviously independent of λ.

These examples motivate the following theorem.

4.3. Theorem (a condition for Gaussian maximizers). *Let G be a degenerate Gaussian kernel with the property that the $n \times n$ real, symmetric matrices A and B in* (1.1) *are both positive definite. If $1 < p \leqq q < \infty$ then $\mathscr{G}$ is bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$. If, additionally, $p < q$ then $\mathscr{G}$ has a maximizer which is a Gaussian function.*

If G is also real then obviously A and B must be positive definite if $\mathscr{G}$ is bounded at all. In this real, degenerate case $\mathscr{G}$ is unbounded when $1 < q < p < \infty$ and $\mathscr{G}$ has no maximizer of any kind when $1 < p = q < \infty$.

Proof. It can be assumed that G is centered and, as in the proof of Theorem 3.3, we can use the fact that A and B are positive definite to change variables so that $G(x, y)$ is brought into the canonical form

$$G(x, y) = \exp\{-(x, x) - (y, y) - 2(x, Ey) - 2\mathrm{i}(x, Hy)\} , \tag{1}$$

where E and H are real matrices and E is also diagonal. In the real case $H = 0$. Since $M = \begin{pmatrix} I & E \\ E & I \end{pmatrix}$ must be positive semidefinite, the eigenvalues $e_1, \ldots, e_n$ of E must be in the interval $[-1, 1]$. Since G is degenerate at least one of the e_i's (say e_1) is $+1$ or -1 and, by changing y to $-y$ if necessary, we can assume that $e_1 = -1$. Thus, $G(x, y)$ contains the factor $\exp\{-(x_1 - y_1)^2\}$.

In the real case, $H = 0$, G in (1) is seen to be a tensor product of operators on $\mathbf{R}^1$, i.e., $G(x, y) = G_1(x_1, y_1) \ldots G_n(x_n, y_n)$. If $p > q$ the operator $\mathscr{G}_1$ corresponding to e_1 is unbounded, as shown in 4.2, so $\mathscr{G}$ is unbounded as well. In case $p \leqq q$ the Minkowski inequality argument in the first part of the proof of Theorem 3.2 (applied sequentially to $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_n$) shows that any maximizer, F, for $\mathscr{G}$ must be of the product form, i.e., $F(y_1, \ldots, y_n) = f_1(y_1) \ldots f_n(y_n)$ and each f_i must be a maximizer for the corresponding $\mathscr{G}_i$. When $p = q$, however, $\mathscr{G}_1$ does not have a maximizer as stated in 4.2 and therefore $\mathscr{G}$ has no maximizer. When $p < q$ we know from 4.2 that each $\mathscr{G}_i$ has a Gaussian maximizer g_i. Since $\prod_i g_i(x_i)$ is a Gaussian function on $\mathbf{R}^n$, the proof for the real case is complete.

For the general case with $p \leqq q$, let $G^0(x, y)$ be the real kernel given by (1) but with H set equal to zero and let $\mathscr{G}^0$ denote the corresponding operator. If $f \in L^p(\mathbf{R}^n) \cap \mathbf{L}^1(\mathbf{R}^n)$ then clearly $\mathscr{R}_{p\to q}(G, f) \leqq \mathscr{R}_{p\to q}(G^0, F)$, where $F \equiv |f|$. Since $\mathscr{G}^0$ is bounded when $p \leqq q$, then $\mathscr{G}$ is also bounded. Referring now to Theorem 4.1 let G_ε be the kernel defined in that proof, i.e., $G_\varepsilon(x, y) = G(x, y)h_\varepsilon(x)h_\varepsilon(y)$ with $h_\varepsilon(x) = \exp\{-\varepsilon(x, x)\}$, and let g_ε denote its unique Gaussian maximizer with $\|g_\varepsilon\|_p = 1$. Let $g_\varepsilon(x) = \mu_\varepsilon \exp\{-(x, J_\varepsilon x) - i(x, K_\varepsilon x)\}$ with J and K real, symmetric and with J_ε positive definite. Define $g_\varepsilon^0(x) = \mu_\varepsilon \exp\{-x, J_\varepsilon x)\}$. Let $\varepsilon \to 0$ through the sequence $\varepsilon = 1/j$ with $j = 1, 2, 3, \ldots$. There is a subsequence of the j's (which we continue to denote by j) such that the eigenvectors of J_ε and K_ε have limits as $j \to \infty$ (because the manifold $O(n)$ is compact). The corresponding eigenvalues of J_ε must be uniformly bounded away from 0 and ∞ since otherwise $\mathscr{R}_{p\to q}(G^0, g_\varepsilon^0)$ will converge to zero, as the following computation shows.

Apart from irrelevant constants, $\|g_\varepsilon^0\|_p = |J_\varepsilon|^{-1/p}$, where $|\cdot|$ denotes the determinant. Also, $\|\mathscr{G}^0(g_\varepsilon^0)\|_q = |J_\varepsilon + I|^{-1}|I - E(J_\varepsilon + I)^{-1}E|^{-1/q}$. Using the fact that $|I - M^TM| = |I - MM^T|$ for any real matrix, M, we have $|I - E(J_\varepsilon + I)^{-1}E| = |I - (J_\varepsilon + I)^{-1/2}E^2(J_\varepsilon + I)^{-1/2}| = |J_\varepsilon + I|^{-1}|J_\varepsilon + I - E^2| \geqq |J_\varepsilon + I|^{-1}|J_\varepsilon|$. Therefore

$$\mathscr{R}_{p\to q}(G^0, g_\varepsilon^0) \leqq |J_\varepsilon|^{1/p-1/q}|J_\varepsilon + I|^{-1/q'} = \prod_{i=1}^{n} (J_\varepsilon^i)^{1/p-1/q}(1 + J_\varepsilon^i)^{-1/q'},$$

where the J_ε^i's are the eigenvalues of J_ε. Since $p < q$ the function $t \mapsto t^{1/p-1/q}(1 + t)^{-1/q'}$ is bounded and goes to zero as $t \to 0$ or $t \to \infty$.

The possibility that $\mathscr{R}_{p\to q}(G^0, g_\varepsilon^0) \to 0$ is not allowed by 4.1(*) and the fact that $\mathscr{R}_{p\to q}(G^0, g_\varepsilon^0) \geqq \mathscr{R}_{p\to q}(G, g_\varepsilon)$. Thus we can pass to a further subsequence such that J_ε has a positive definite limit J as $\varepsilon \to 0$. This implies that μ_ε also has a finite, nonzero limit μ.

The eigenvalues of K_ε must also stay bounded away from infinity for otherwise g_ε would tend weakly to zero in $L^p(\mathbf{R}^n)$ and then the function $\mathscr{G}(g_\varepsilon)$ would tend to zero pointwise. (This is so because the function $y \mapsto G(x, y)\exp\{-\frac{1}{2}(y, Jy)\}$ is in $L^{p'}(\mathbf{R}^n)$ for each x.) But $\mathscr{G}(g_\varepsilon)$ is bounded above pointwise by $\mathscr{G}^0(g_\varepsilon^0)$, and the pointwise convergence to zero would imply by dominated convergence that $\mathscr{G}(g_\varepsilon)$ converges to zero in $L^q(\mathbf{R}^n)$ norm. Thus, by passing to a further subsequence, J_ε and

K_ε have limits J and K. From this it follows that g_ε converges strongly in $L^p(\mathbf{R}^n)$ norm to $g(x) = \mu \exp\{-(x, Jx) - i(x, Kx)\}$.

The Gaussian function g is the desired maximizer for $\mathscr{G}$. First note that $h_\varepsilon g \to g$ in $L^p(\mathbf{R}^n)$ norm as $\varepsilon \to 0$. Also $g_\varepsilon \to g$, and thus we can write $h_\varepsilon g_\varepsilon = g + \Delta_\varepsilon$ with $\delta_\varepsilon \equiv \|\Delta_\varepsilon\|_p \to 0$ as $\varepsilon \to 0$. Then, since $\mathscr{G}$ is bounded,

$$C_{p\to q}(G) \geqq \mathscr{R}_{p\to q}(G, g) \geqq \mathscr{R}_{p\to q}(G, h_\varepsilon g_\varepsilon) - C_{p\to q}\delta_\varepsilon .$$

Taking the limit $\varepsilon \to 0$,

$$C_{p\to q}(G) \geqq \mathscr{R}_{p\to q}(G, g) \geqq \limsup_{\varepsilon\to 0} \mathscr{R}_{p\to q}(G, h_\varepsilon g_\varepsilon) .$$

But by Eq. (5) of the proof of Theorem 4.1, this latter limit equals $C_{p\to q}(G)$. □

4.4. Remarks and conjectures. Formula 4.1(∗) gives the sharp bound. The question that is incompletely resolved here is whether there is a Gaussian maximizer in the degenerate case or, indeed, any maximizer at all. In the cases of most interest (e.g., Nelson's kernel of Sect. I and the Fourier transform) the existence of a Gaussian maximizer can easily be verified by simple computation. The general case is algebraically complex, although Theorem 4.3 does give a criterion for a Gaussian maximizer and it completely settles the case of real Gaussian kernels. Indeed, as shown in 4.2, a maximizer need not exist even if $\mathscr{G}$ is bounded.

The examples given here lead to the following *conjectures.*

(1) If there is a maximizer for cases (A), (B) or (C) of Sect. I then there is a Gaussian maximizer.
(2) There is a maximizer in these cases if and only if the unique Gaussian maximizer g_ε for the mollified kernel $G_\varepsilon(x, y) = G(x, y)h_\varepsilon(x)h_\varepsilon(y)$ defined in the proof of Theorem 4.1 has a strong limit g in $L^p(\mathbf{R}^n)$ as $\varepsilon \to 0$.

Maximizers need not be unique, as shown in 4.2, but *if there is any Gaussian maximizer for $p < q$ then every maximizer is a Gaussian.* This is Theorem 4.5, and it completely settles the Fourier transform case, for example. (Note that when $p = q = 2$, every function in $L^2(\mathbf{R}^n)$ is a maximizer for the Fourier transform and thus there is at least one case in which there are maximizers that are not Gaussians.)

Theorem 4.5 also completely settles the real Case (A) because, by Theorem 4.3, no maximizer exists in this case when $p \geqq q$ and a Gaussian maximizer does exist when $p < q$.

4.5. Theorem (when $p < q$, a Gaussian maximizer implies all maximizers are Gaussians). *Let $1 < p < q < \infty$ and let G be a degenerate Gaussian kernel. Assume that $\mathscr{G}$ is a bounded operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ and that g is a Gaussian function that is a maximizer for $\mathscr{G}$. If $f \in L^p(\mathbf{R}^n)$ is another maximizer for $\mathscr{G}$ then f is also a Gaussian (but f is not necessarily proportional to g and f is not necessarily centered even if G is).*

Proof. Step 1. According to Lemmas 2.2 and 2.3 it can be assumed without loss of generality that both G and g are centered. As in the proof of Theorem 3.4, we study the kernel $G^{(2)} = G \otimes G$. For $F \in L^p(\mathbf{R}^{2n}) \cap L^1(\mathbf{R}^{2n})$ the inequalities (1)–(5) there are valid and we conclude that $C_{p\to q}(G^{(2)}) = (C_{p\to q})^2$, where $C_{p\to q} \equiv C_{p\to q}(G)$.

Step 2. If $f \in L^p(\mathbf{R}^n)$ is a maximizer for $\mathscr{G}$ then, using $O(2)$ invariance again,

$$F(y_1, y_2) \equiv f\left(\frac{y_1 + y_2}{\sqrt{2}}\right) g\left(\frac{y_1 - y_2}{\sqrt{2}}\right) \tag{1}$$

is obviously a maximizer for $\mathscr{G}^{(2)}$ if f is also in $L^1(\mathbf{R}^n)$, in which case $F \in L^1(\mathbf{R}^{2n})$. If $f \notin L^1(\mathbf{R}^n)$ consider the mollified function $F_j(y_1, y_2) = F(y_1, y_2) \exp\{-(y_1 + y_2, y_1 + y_2)/j\}$ for $j = 1, 2, \ldots$, which is in $L^1(\mathbf{R}^{2n})$. Clearly $F_j \to F$ strongly in $L^p(\mathbf{R}^{2n})$ as $j \to \infty$. The function $\mathscr{G}^{(2)} F_j$ can be computed as a $\mathrm{d}y_1 \mathrm{d}y_2$ integral of $G^{(2)} F_j$ and the result (using the $O(2)$ invariance of $G^{(2)}$ and a change of variables) is

$$(\mathscr{G}^{(2)} F_j)(x_1, x_2) = (\mathscr{G} f_j)\left(\frac{x_1 + x_2}{\sqrt{2}}\right) \cdot (\mathscr{G} g)\left(\frac{x_1 - x_2}{\sqrt{2}}\right)$$

with $f_j(y) = f(y) \exp\{-2(y, y)/j\}$. Now the q^{th} power integral of $\mathscr{G}^{(2)} F_j$ can be computed by changing variables again and the result is $\| \mathscr{G}^{(2)} F_j \|_q = \| \mathscr{G} f_j \|_q \| \mathscr{G} g \|_q$. However $\| \mathscr{G} f_j \|_q \to \| \mathscr{G} f \|_q = C_{p \to q} \| f \|_p$ as $j \to \infty$ since $f_j \to f$ in $L^p(\mathbf{R}^n)$ norm, and we conclude that $\| \mathscr{G}^{(2)} F \|_q = \lim_{j \to \infty} \| \mathscr{G}^{(2)} F_j \|_q$ (by definition) $= (C_{p \to q})^2 \| F \|_p$, so that F is indeed a maximizer for $\mathscr{G}^{(2)}$.

Step 3. Since g is a Gaussian, it is obvious that the function $z \mapsto F(z, y)$ is in $L^p(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ for each y and therefore that

$$K(x, y) \equiv \int G(x, z) F(z, y) \mathrm{d}z \tag{2}$$

is well defined for each x and y in $\mathbf{R}^n$. Since $\mathscr{G}$ is a bounded operator, the function $x \mapsto K(x, y)$ is in $L^q(\mathbf{R}^n)$ for each y. We now assert that the function $y \mapsto K(x, y)$ is in $L^p(\mathbf{R}^n)$ for almost every $x \in \mathbf{R}^n$ and that this function satisfies

$$\{ \int [\int |K(x, y)|^q \mathrm{d}x]^{p/q} \mathrm{d}y \}^{q/p} = \int \{ \int |K(x, y)|^p \mathrm{d}y \}^{q/p} \mathrm{d}x \,, \tag{3}$$

with the understanding that both sides of (3) are finite. Formally, this assertion is a consequence of inequality (3) → (4) in the proof of Theorem 3.4 and the fact that all the inequalities (1)–(5) must be equalities since $F(y_1, y_2)$ is a maximizer for $\mathscr{G}^{(2)}$. If $F \in L^1(\mathbf{R}^{2n})$ this would be correct, but if $F \notin L^1(\mathbf{R}^{2n})$ a proof is needed.

Set $F_j(y_1, y_2) = F(y_1, y_2) \exp\{-(y_2, y_2)/j\}$ for $j = 1, 2, \ldots$. Clearly $F_j \in L^1(\mathbf{R}^{2n})$ and $F_j \to F$ strongly in $L^p(\mathbf{R}^{2n})$ as $j \to \infty$. (Note that this F_j is not the same one as in step 2.) Let $K_j(x, y)$ be as in (2) with F replaced by F_j, so that $K_j(x, y) = K(x, y) \exp\{-(y, y)/j\}$. The inequalities (1)–(5) in the proof of Theorem 3.4 are then valid with F replaced by F_j. As $j \to \infty$ the left side of these inequalities, namely $\| \mathscr{G}^{(2)} F_j \|_q^q$, converges to $\| \mathscr{G}^{(2)} F \|_q^q = (C_{p \to q})^{2q} \| F \|_p^q \equiv (C_{p \to q})^q Z$ since F is a maximizer. Likewise, the right side, namely $(C_{p \to q})^{2q} \| F_j \|_p^q$ also converges to $(C_{p \to q})^q Z$ since $F_j \to F$. Therefore the numbers

$$B_j \equiv \{ \int [\int |K_j(x, y)|^q \mathrm{d}x]^{p/q} \mathrm{d}y \}^{q/p} - \int \{ \int |K_j(x, y)|^p \mathrm{d}y \}^{q/p} \mathrm{d}x \tag{4}$$

(which are nonnegative by Minkowski's inequality) must converge to zero as $j \to \infty$. Moreover, each term in B_j is bounded by $(C_{p \to q})^q \| F_j \|_p^q < Z$, and each term converges to Z as $j \to \infty$ (because of inequalities (1)–(5)). The first term in B_j is

$$A_j \equiv \{ \int [\int |K(x, y)|^q \mathrm{d}x]^{p/q} \exp\left\{ -\frac{p}{j}(y, y) \right\} \mathrm{d}y \Big\}^{q/p}$$

and, by the monotone convergence theorem, A_j converges to $A \equiv$ (the left side of (3)). Therefore $A = Z$. The second term in B_j is

$$D_j \equiv \int \left\{ \int |K(x, y)|^p \exp\left\{ -\frac{p}{j}(y, y) \right\} \mathrm{d}y \right\}^{q/p} \mathrm{d}x \,.$$

The inner integral (call it $E_j(x)$) converges (by monotone convergence) to $E(x) \equiv \int |K(x, y)|^p \, dy$. The function E is measurable since it is the monotone limit of measurable functions E_j. Then $\int \{E_j\}^{q/p}$ converges to $\int \{E\}^{q/p}$ by monotone convergence, so D_j converges to the right side of (3). But, as stated above, D_j also converges to Z, so the two sides of (3) are equal and $E(x)$ is finite for almost every x, as asserted.

Step 4. Since $q > p$, the strong form of Minkowski's inequality and the equality in (3) implies the existence of measurable functions α and $\beta: \mathbf{R}^n \to [0, \infty)$ such that

$$|K(x, y)| = \alpha(x)\beta(y) \tag{5}$$

for almost every x and y in $\mathbf{R}^n$. Writing $G(x, y) = \exp\{-(x, Ax) - (y, By) - 2(x, Dy)\}$ as usual (with A and B real, symmetric, positive definite), and writing $g(y) = \exp\{-(y, Jy)\}$ (with J symmetric and $\mathrm{Re}(J)$ positive definite) a simple computation gives

$$K(x, y) = \exp\{-(x, Ax) + (D^T x, (B + J)^{-1} D^T x) - (y, Jy)\} Q((B + J)y - D^T x) \tag{6}$$

with $Q: \mathbf{C}^n \to \mathbf{C}$ given by

$$Q(w) = \exp\{-(w, (B + J)^{-1} w)\} \int f\left(\frac{z}{\sqrt{2}}\right) \exp\{-(z, (B + \tfrac{1}{2} J)z) + 2(z, w)\} \, dz \, . \tag{7}$$

Evidently Q is an entire analytic function of order at most 2. Define the function $M: \mathbf{R}^{2n} \to \mathbf{C}$ by $M(x, y) = Q((B + J)y - D^T x)$. Plainly, since Q is entire M has an extension to an entire analytic function from $\mathbf{C}^{2n}$ to $\mathbf{C}$; call this extension N. The function $N^*: \mathbf{C}^{2n} \to \mathbf{C}$ defined by $N^*(x, y) = \overline{N(\bar{x}, \bar{y})}$ for x and $y \in \mathbf{C}^n$ is also entire analytic, and thus $P \equiv NN^*$ is entire analytic as well. It is also true that $P(x, y) = |M(x, y)|^2$ when x and y are in $\mathbf{R}^n$. From (5) and (6),

$$P(x, y) = \gamma(x)\delta(y) \tag{8}$$

for almost every x and y in $\mathbf{R}^n$, and where γ and $\delta: \mathbf{R}^n \to [0, \infty)$ are the measurable functions given by $\gamma(x) = \alpha(x)^2 \exp\{2(x, Ax) - 2\mathrm{Re}((D^T x, (B + J)^{-1} D^T x))\}$ and $\delta(y) = \beta(y)^2 \exp\{2\mathrm{Re}((y, Jy))\}$. If y_0 is a value of y such that $\delta(y_0) \neq 0$ and such that (8) holds for almost every x, we see by substituting this y_0 in (8) that γ has an extension to an entire analytic function. Likewise, δ has an extension. Thus (8) holds for *every* x and y in $\mathbf{C}^n$ (because if two entire functions agree almost everywhere on $\mathbf{R}^n \times \mathbf{R}^n$ then they agree on all of $\mathbf{C}^n \times \mathbf{C}^n$).

Now suppose that $\gamma(x_0) = 0$ for some x_0 in $\mathbf{C}^n$. Then, by (8), $P(x_0, y) = 0$ for every $y \in \mathbf{C}^n$, which implies that for each y either (*i*) $N(x_0, y) = 0$ or (*ii*) $N^*(x_0, y) = 0$. This, in turn, means that for each $y \in \mathbf{C}^n$ either (i) $N(x_0, y) \equiv Q((B + J)y - D^T x_0) = 0$ or (ii) $N(\bar{x}_0, \bar{y}) \equiv Q((B + J)\bar{y} - D^T \bar{x}_0) = 0$. Necessarily, either case (i) holds for all y in some set $S \subset \mathbf{C}^n$ of positive $2n$-dimensional Lebesgue measure $\mathscr{L}^{2n}$ or case (ii) holds in some set S of positive $\mathscr{L}^{2n}$ measure. As y ranges over S both $(B + J)y$ and $(B + J)\bar{y}$ range over sets of positive $\mathscr{L}^{2n}$ measure (because $\mathrm{Re}(B + J)$ is positive definite and therefore $Rank(B + J) = n$). An analytic function that vanishes on a set of positive $\mathscr{L}^{2n}$ measure vanishes identically, and thus Q would vanish identically if $\gamma(x_0) = 0$. This contradicts the

fact that $K(x, y)$ is not identically zero. Thus, the assumption that $\gamma(x_0) = 0$ is not possible, and it will be assumed henceforth that $\gamma(x) \neq 0$ for all $x \in \mathbf{C}^n$.

Define the set $A = \{y \in \mathbf{R}^n : \delta(y) \neq 0\} \subset \mathbf{R}^n$. This set A has positive n-dimensional Lebesgue measure $\mathscr{L}^n$, for otherwise $K(x, y) = 0$, $\mathscr{L}^{2n}$ almost everywhere. (In fact $\mathscr{L}^n(\mathbf{R}^n \sim A) = 0$ because δ is analytic and δ does not vanish identically, but this fact is not needed.) For $y \in A$, the function $Z_y : \mathbf{C}^n \to \mathbf{C}$ defined by $Z_y(x) = K(x, y)$ is entire analytic of order at most 2 and never zero (because $\gamma(x)$ is never zero). Then Z_y has the form

$$Z_y(x) = K(x, y) = \exp\{-(x, T_y x) - (R_y, x) + \mu_y\}, \tag{9}$$

where T_y is a complex, symmetric matrix, $R_y \in \mathbf{C}^n$ and $\mu_y \in \mathbf{C}$ (all of which depend on y). I thank Eric Carlen for the simple proof of this fact, which is that Z_y, being zero free, has an entire analytic logarithm, i.e., $Z_y = \exp\{H_y\}$. Then, since Z_y has order at most 2, $|H_y(x)|$ is bounded above by (const.) $|x|^2$. By a well known argument using Cauchy's integral formula, H_y must be a polynomial whose order is at most 2, i.e., Z_y has the form stated in (9).

Step 5. As noted in step 2, the function $x \mapsto K(x, y)$ is in $L^q(\mathbf{R}^n)$ for almost every $y \in \mathbf{R}^n$. By (4) $\to$ (5) of Theorem 3.4, the function $z \mapsto F(z, y)$ (which is in $L^p(\mathbf{R}^n)$ for almost every y) must be a maximizer of $\mathscr{R}_{p\to q}$ for almost every y. (Note that $z \mapsto F(z, y)$ cannot be the zero function for any y since g never vanishes.) Thus there is at least one point $y_0 \in \mathbf{R}^n$ such that $\delta(y_0) \neq 0$ and (9) holds and such that $z \mapsto F(z, y_0)$ is a maximizer in $L^p(\mathbf{R}^n)$. Fix this y_0 henceforth and denote the matrix in (9) simply by T. There is then a function $h \in L^{q'}(\mathbf{R}^n)$ with $\|h\|_{q'} = 1$ such that

$$\int h(x)K(x, y_0)\mathrm{d}x = \|K(\cdot, y_0)\|_q = C_{p\to q}\|F(\cdot, y_0)\|_p . \tag{10}$$

Since $K(\cdot, y_0) \in L^q(\mathbf{R}^n)$, the matrix T must satisfy $\mathrm{Re}(T)$ is positive definite and therefore $K(\cdot, y_0)$ is a Gaussian. The optimum h satisfies $h(x) = (\text{const.})\, |K(x, y_0)|^q / K(x, y_0)$ for $x \in \mathbf{R}^n$ and therefore h is also a Gaussian (and hence $h \in L^1(\mathbf{R}^n)$). As remarked in step 3, $F(\cdot, y_0)$ is in $L^1(\mathbf{R}^n)$. Therefore the function $(x, y) \mapsto h(x)G(x, y)F(y, y_0)$ is in $L^1(\mathbf{R}^{2n})$ and Fubini's theorem can be applied to (10). Thus,

$$\int h(x)K(x, y_0)\mathrm{d}x = \int \{\int h(x)G(x, z)\mathrm{d}x\} F(z, y_0)\mathrm{d}z . \tag{11}$$

Since h is a Gaussian, the inner integral in (11) (call it $k(z)$) is also a Gaussian. Since $F(\cdot, y_0)$ is a maximizer, $F(z, y_0) = (\text{const.})|k(z)|^{p'}/k(z) \equiv r(z)$ for almost every $z \in \mathbf{R}^n$. Clearly r is a Gaussian and, by (1)

$$f\left(\frac{z + y_0}{\sqrt{2}}\right) g\left(\frac{z - y_0}{\sqrt{2}}\right) = r(z) \tag{12}$$

for almost every $z \in \mathbf{R}^n$. Setting $z = w - y_0$, (12) yields $f(w/\sqrt{2}) = r(w - y_0)/g((w - 2y_0)/\sqrt{2})$, which is a Gaussian (in w) as asserted in the theorem. □

V. Gaussian kernels from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^m)$

This section consists essentially of a simple remark, but it can be a useful one in applications, e.g., in [L1]. Let G be a Gaussian kernel on $\mathbf{R}^m \times \mathbf{R}^n$ with $m \neq n$, i.e., $G(x, y)$ is given by (1.1) with A $m \times m$ symmetric, B $n \times n$ symmetric, D $m \times n$ and

$L \in C^{m+n}$, and with M in (1.3) a positive semidefinite $(m+n) \times (m+n)$ matrix. Evidently Lemmas 2.1, 2.2 and 2.3 continue to hold in this case, and it can be assumed without loss of generality that A and B are real and $L = 0$. The linear operator $\mathscr{G}$ from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^m)$ and the norm $C_{p\to q}(G)$ are defined, *mutatis mutandis*, as in Sect. I. The remark is the following.

5.1. Theorem (extension to $m \neq n$). *Let G be a Gaussian kernel on $\mathbf{R}^m \times \mathbf{R}^n$ as defined above. Then all the preceding theorems and lemmas in this paper holds, mutatis mutandis, in this more general case.*

Proof. Suppose $m < n$ and extend G to a Gaussian kernel, $\tilde{G}$, on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$\tilde{G}(x, y) = h(x_1)G(x_2, y)$$

where $x \in \mathbf{R}^n$ is written as (x_1, x_2) with $x_1 \in \mathbf{R}^{n-m}$ and $x_2 \in \mathbf{R}^m$, and where $h(x_1) \equiv \exp\{-(x_1, x_1)\}$. Let $\tilde{\mathscr{G}}$ be the corresponding operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$. Note that $\tilde{G}$ has the same properties as G, i.e., the degeneracy or nondegeneracy of $\tilde{G}$ is the same as that of G; $\tilde{G}$ is in Case (A), (B) or (C) if G is; the $n \times n$ matrix $\begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}$ is positive definite if and only if A is. Also, $\tilde{\mathscr{G}}$ is unbounded if $\mathscr{G}$ is, and it will be assumed henceforth that $\mathscr{G}$ is bounded.

If $f \in L^p(\mathbf{R}^n)$ then evidently, as functions in $L^q(\mathbf{R}^n)$, $(\tilde{\mathscr{G}}f)(x) = h(x_1)(\mathscr{G}f)(x_2)$ and thus $\|\tilde{\mathscr{G}}f\|_{L^p(\mathbf{R}^n)} = \|h\|_{L^q(\mathbf{R}^{n-m})}\|\mathscr{G}f\|_{L^q(\mathbf{R}^m)}$. This proves that $C_{p\to q}(\tilde{G}) = C_{p\to q}(G)\|h\|_{L^q(\mathbf{R}^{n-m})}$ and that f is a maximizer for $\tilde{\mathscr{G}}$ if and only if f is a maximizer for $\mathscr{G}$. This concludes the $m < n$ case.

If $m > n$ duality can be used: $C_{p\to q}(G) = C_{q'\to p'}(G^T)$ where $G^T(x, y) = G(y, x)$. This changes the $m > n$ case into the $m < n$ case and, since all the theorems in this paper are "duality invariant", the $m > n$ case is proved. □

Remark. Clearly the proof of Theorem 5.1 is such that if other cases with $m = n$ are settled in the future then Theorem 5.1 for $m \neq n$ holds for those cases as well.

VI. Multilinear forms in the real case and Young's inequality

After Sects. I to V were completed, Eric Carlen suggested that the same methods should yield similar results for real multilinear forms. Indeed this is so and the proof is outlined here (the omitted details are merely a repetition of those given before). Some remarks about the complex case will also be made here. Finally, Theorem 6.2 contains an application of the result in Sect. 6.1 for real multilinear forms: The truly multidimensional generalization of Young's inequality, which was surmised in [BL, p. 162], will be proved.

6.1. **Multilinear forms.** For $i = 1, 2, \ldots, K$ let n_i be a positive integer and let x_i denote a point in $\mathbf{R}^{n_i}$. The point $X = (x_1, \ldots, x_K)$ denotes a point in $\mathbf{R}^N$ with $N = \sum_{i=1}^K n_i$. Let $G(X)$ be a "Gaussian kernel", i.e.,

$$G(X) = \exp\left\{ -\sum_{i=1}^K \sum_{j=1}^K (x_i, A_{ij}x_j) + 2(L, X) \right\},$$

where A_{ij} is a $n_i \times n_j$ matrix with $A_{ij} = A_{ji}^T$, and where $L \in \mathbf{C}^N$. The $N \times N$ symmetric matrix A is the matrix whose blocks are the A_{ij}'s and G is said to be **nondegenerate** if $M \equiv \mathrm{Re}(A)$ is positive definite. Otherwise $M \geqq 0$ and G is **degenerate**.

Let $P = (p_1, \ldots, p_K)$ satisfy $1 < p_i < \infty$ for each i. The **multilinear form** is

$$\mathscr{G}(f_1, \ldots, f_K) = \int G(x_1, \ldots, x_K) \prod_{i=1}^{K} f_i(x_i) \mathrm{d}x_1 \ldots \mathrm{d}x_K , \tag{1}$$

where the integration is over $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \times \ldots \times \mathbf{R}^{n_K}$ and each $f_i \in L^{p_i}(\mathbf{R}^{n_i})$. The problem is to evaluate

$$C_P = \sup \mathscr{G}(f_1, \ldots, f_K) \tag{2}$$

where the supremum is over f_i's with $\|f_i\|_{p_i} = 1$. As before, if G is degenerate we have to take $f_i \in L^{p_i}(\mathbf{R}^{n_i}) \cap L^1(\mathbf{R}^{n_i})$ and then take limits.

The cases treated in Sects. I to V correspond to $K = 2$ with $p_1 = p$ and $p_2 = q'$. The case $K = 1$ is trivial—by Hölder's inequality.

Lemma 2.1 is easily generalized to the complex, nondegenerate multilinear case; the details are left to the reader. The conclusion of Lemma 2.1 holds for each f_i in a maximizing set $(f_1, \ldots, f_K)$. The conclusions (a), (b) and (c) follow by fixing all the f_j's with $j \neq i$ and then investigating the dependence of $\mathscr{G}(f_1, \ldots, f_K)$ on f_i.

Lemma 2.2 obviously carries through as well; that is A_{ii} can be assumed to be real and G can be assumed to be centered, i.e., $L = 0$. Likewise Lemma 2.3 carries through: When G is centered (i.e., $L = 0$) and when the supremum in (2) is restricted to Gaussian functions f_i, then each f_i can be taken to be centered and, in the nondegenerate case, each f_i *must be* centered.

Let us now turn to the real case, i.e., each A_{ij} is real and $L = 0$. *Theorem* 3.2 *for the nondegenerate case carries through for every choice* of P. The maximizing K-tuple $(f_1, \ldots, f_K)$ is unique (up to multiplicative constants) and each $f_i(x) = \exp\{-(x, J_i x)\}$ with J_i being real and positive definite. To prove that f_1, say, has this property we write (with $q = p_1'$)

$$C_P = \sup \|\tilde{\mathscr{G}}(f_2, \ldots, f_K)\|_q$$

where the supremum is on $f_2, \ldots, f_K$ with $\|f_j\|_{p_j} = 1$ and where

$$\tilde{\mathscr{G}}(f_2, \ldots, f_K)(x_1) = \int G(x_1, \ldots, x_K) \prod_{j=2}^{K} f_j(x_j) \mathrm{d}x_2 \ldots \mathrm{d}x_K .$$

As before, we replace $\tilde{\mathscr{G}}$ by $\tilde{\mathscr{G}}^{(2)} = \tilde{\mathscr{G}} \otimes \tilde{\mathscr{G}}$ and $f_2(x_2), \ldots, f_K(x_K)$ by $F_2(x_2, y_2), \ldots, F_K(x_K, y_K)$ with $F_j \in L^{p_j}(\mathbf{R}^{2n_j})$. To imitate the inequalities (1)–(5) in Theorem 3.2, define

$$K(x_1, y_2, \ldots, y_K) = \int G(x_1, x_2, \ldots, x_K) \prod_{j=2}^{K} F_j(x_j, y_j) \mathrm{d}x_2 \ldots \mathrm{d}x_K .$$

Then, proceeding as in (1)–(5) (and with the F_j nonnegative for the same reason as before)

$$\begin{aligned}
\|\tilde{\mathscr{G}}^{(2)}(F_2, \ldots, F_K)\|_q^q &= \int [\int G(y_1, \ldots, y_K) K(x_1, y_2, \ldots, y_K) \mathrm{d}y_2 \ldots \mathrm{d}y_K]^q \mathrm{d}x_1 \mathrm{d}y_1 \\
&\leqq \int \{\int G(y_1, \ldots, y_K) [\int K(x_1, y_2, \ldots, y_K)^q \mathrm{d}x_1]^{1/q} \\
&\quad \times \mathrm{d}y_2 \ldots \mathrm{d}y_K\}^q \mathrm{d}y_1 \\
&\leqq (C_P)^q \int \{\int G(y_1, \ldots, y_K) \prod_{j=2}^{K} h_j(y_j) \mathrm{d}y_2 \ldots \mathrm{d}y_K\}^q \mathrm{d}y_1 \\
&\quad \text{(with } h_j(y) = [\int F_j(x, y)^{p_j} \mathrm{d}x]^{1/p_j}) \\
&\leqq (C_P)^{2q} \prod_{j=2}^{K} \|F_j\|_{p_j}^q .
\end{aligned}$$

As before, Minkowski's inequality implies that

$$K(x_1, y_2, \ldots, y_K) = A(x_1)E(y_2, \ldots, y_K)\,,$$

which then implies that

$$\tilde{\mathcal{G}}^{(2)}(F_2, \ldots, F_K)(x_1, y_1) = A(x_1)Z(y_1)\,.$$

However, $\tilde{\mathcal{G}}^{(2)}(F_2, \ldots, F_K) = F_1^{p_1 - 1}$, and hence F_1 is a product function. The rest of the proof is identical to the proof of Theorem 3.2.

By taking limits, the analogue of Theorem (4.1) hold in the degenerate case whenever it is known that the nondegenerate case has a Gaussian maximizer K-tuple. *In particular, Theorem* 4.1 *holds in the real case.* The analogue of 4.1(*) is that C_P is given by (2) with the supremum restricted to centered Gaussian functions. Likewise, Theorem 4.3 extends to the multilinear case under the same assumption about the nondegenerate case; the analogous hypothesis is that each A_{ii} is positive definite. *In particular Theorem* 4.3 *holds in the real case.*

These results can be used to derive the sharp constants in the fully multidimensional generalized Young's inequality. Recall that Young's original inequality states that if $f \in L^p(\mathbf{R}^n)$ and $g \in L^q(\mathbf{R}^n)$ then $f * g \in L^r(\mathbf{R}^n)$ with $1/p + 1/q = 1 + 1/r$; here $*$ denotes convolution. The sharp constant in this inequality was derived simultaneously by Beckner [B1, B2] and by Brascamp and Lieb [BL]. Another way to state the inequality is that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} h(x)f(x-y)g(y)\mathrm{d}x\mathrm{d}y \leqq C\|h\|_{r'}\|f\|_p\|g\|_q \tag{3}$$

with $1/p + 1/q + 1/r' = 2$. The Beckner, Brascamp-Lieb result is that C can be determined by restricting f, g and h to be Gaussian functions. (These, in fact, are the only maximizers, as shown in [BL].)

Young's inequality (3) was generalized in several ways in [BL]. The first way is to allow an arbitrary number of functions $f_1, \ldots, f_K$ instead of merely three as in (3). These are functions from $\mathbf{R}^n$ to $\mathbf{C}$ and $f_j \in L^{p_j}(\mathbf{R}^n)$. This is Theorem 7 of [BL]. The integration is then over $(\mathbf{R}^m)^n$ and the arguments of the f_j's are taken to be $((a_1^j, x_1), \ldots, (a_n^j, x_n)) \in \mathbf{R}^n$, where $a_i^j \in \mathbf{R}^m$ are specified vectors and $x_i \in \mathbf{R}^m$. Unfortunately, this is not a fully mn-dimensional generalization of the $n = 1$ result because $\mathbf{R}^{mn}$ is split unnaturally into $(\mathbf{R}^m)^n$. Following Theorem 7 in [BL] we asked whether the full generalization is possible and Theorem 6.2 below gives it.

A second generalization was the incorporation of a fixed Gaussian function in the integral, as in Theorem 6 of [BL]. Again, the Gaussian in [BL] was completely general when $n = 1$, but not otherwise. In Theorem 6.2 it is completely general.

6.2. Theorem (fully generalized Young's inequality). *Fix $K > 1$, $n_1, \ldots, n_K$ and $p_1, \ldots, p_K > 1$ as before. Let $M \geqq 1$ be an integer and let B_i (for $i = 1, \ldots, K$) be a linear mapping from $\mathbf{R}^M$ to $\mathbf{R}^{n_i}$. For nonnegative functions $f_1, \ldots, f_K$, with $f_i \in L^{p_i}(\mathbf{R}^{n_i})$ consider*

$$I(f_1, \ldots, f_K) = \int_{\mathbf{R}^M} \prod_{i=1}^{K} f_i(B_i x)\mathrm{d}x\,. \tag{1}$$

More generally, let $g: \mathbf{R}^M \to \mathbf{R}^+$, $g(x) = \exp\{-(x, Jx)\}$, be a fixed, centered, real Gaussian function and consider

$$I_g(f_1, \ldots, f_K) = \int_{\mathbf{R}^M} \prod_{i=1}^{K} f_i(B_i x)g(x)\mathrm{d}x\,. \tag{2}$$

Let

$$C_g \equiv \sup_{f_1,\ldots,f_K} \{I_g(f_1,\ldots,f_K): \|f_i\|_{p_i} = 1\} \tag{3}$$

and similarly for C (with I_g replaced by I). Then

$$C_g = \sup\{I_g(f_1,\ldots,f_K): f_1,\ldots,f_K \text{ are real, centered Gaussian functions with } \|f_i\|_{p_i} = 1\}\,, \tag{4}$$

and similarly for C.

Proof. Suppose the theorem is false and that the right side of (4) (call it D_g) is strictly smaller than C_g. (Alternatively, $D < C$.) Then there are nonnegative summable functions that are *not all* Gaussians, $f_1,\ldots,f_K$ of unit L^{p_i} norm, such that $I_g(f_1,\ldots,f_K) > D_g$ (or $I(f_1,\ldots,f_K) > D$).

Consider the functions $f_l^{(l)}: \mathbf{R}^{n_i} \to \mathbf{R}^+$ given by $f_l^{(l)} \equiv f_i * g_l^{(l)}$ for l a positive integer, where $g_l^{(l)}(x) \equiv (l/\pi)^{n_i/2} \exp\{-l(x,x)\}$ is an $L^1(\mathbf{R}^{n_i})$ normalized Gaussian function. We note that $\|f_l^{(l)}\|_{p_i} \leqq 1$ and that $f_l^{(l)} \to f_i$ in $L^{p_j}(\mathbf{R}^{n_j})$ as $l \to \infty$. By passing to a subsequence (henceforth still denoted by l) we can assume that $f_l^{(l)}(x) \to f_i(x)$ for almost every x in $\mathbf{R}^{n_i}$.

Evidently we can assume that $M \geqq \max\{n_1,\ldots,n_K\}$ and that the rank of B_i is n_i for all i. Otherwise, I or I_g involves knowledge of some f_i only on a hyperplane in $\mathbf{R}^{n_i}$ and this means that I or I_g can be made arbitrarily large (with all f_i's being Gaussian functions) while preserving $\|f_i\|_{p_i} = 1$; the theorem would then be true in this case because both sides of (4) would be infinite. Similarly, the mapping $W \equiv J + \sum_{i=1}^K B_j^* B_j$ (with $*$ denoting adjoint) from $\mathbf{R}^M$ to $\mathbf{R}^M$ is positive definite; otherwise I_g can again be made arbitrarily large with Gaussian f_i's. A similar condition holds for I with $J = 0$. Since B_i is linear and has full rank n_i, the almost everywhere pointwise convergence of $f_l^{(l)}$ to f_i in $\mathbf{R}^{n_i}$ implies that $f_l^{(l)}(B_i x) \to f_i(B_i x)$ for almost every x in $\mathbf{R}^M$.

By Fatou's lemma

$$C_g' \equiv \liminf_{l\to\infty} I_g(f_l^{(l)},\ldots,f_K^{(l)}) \geqq I_g(f_1,\ldots,f_K) > D_g \tag{5}$$

and similarly for C' (with I in place of I_g). By Fubini's theorem, however,

$$I_g(f_1^{(l)},\ldots,f_K^{(l)}) = \int_{\mathbf{R}^N} G_g^{(l)}(y_1,\ldots,y_K) \prod_{i=1}^K f_i(y_i)\,\mathrm{d}y_1 \ldots \mathrm{d}y_K\,. \tag{6}$$

Here $N = \sum_{i=1}^K n_i$ as in Sect. 6.1, $y_i \in \mathbf{R}^{n_i}$, and $G_g^{(l)}$ is the centered Gaussian kernel

$$G_g^{(l)}(y_1,\ldots,y_K) = \int_{\mathbf{R}^M} \prod_{i=1}^K g_l^{(l)}(B_i x - y_i) g(x)\,\mathrm{d}x\,. \tag{7}$$

Similarly, (6) and (7) hold for I in place of I_g by deleting the g. (Note: Because W is positive definite, the integral in (7) is always finite.)

The number C_g' defined in (5) is either finite or infinite. In either case, there is some finite integer k such that $C_g'' \equiv I_g(f_1^{(k)},\ldots,f_K^{(k)}) > D_g$. However, by (6) we see that C_g'' is a multilinear form as in 6.1 (1). Such a form has the property, as we have seen in Section 6.1, that its supremum over f_i's with $\|f_i\|_{p_i} = 1$ is equal to its supremum over real, centered Gaussian functions. But if we set all the f_i's equal to Gaussian functions we have that $f_l^{(k)}$'s are also Gaussian functions and $\|f_l^{(k)}\|_{p_i} \leqq 1$.

This means that $C_g'' \leqq D_g$, and this is a contradiction. The same proof holds for I in place of I_g. □

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Chamber systems, geometries and parabolic systems whose diagram contains only bonds of strength 1 and 2

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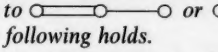
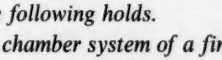
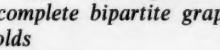
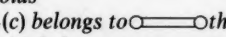
The aim of this paper is to study a certain type of chamber systems and give application to the classification of geometries whose diagram is a Coxeter diagram. In this sense this paper is a successor of [13]. For notations and definitions concerning chamber systems see [13] or [19]. For convenience of the reader we recall two definitions which are of importance for this paper. Let $\mathscr{C} = (\mathscr{C}, (\mathfrak{P}_i)_{i\in I})$ be a chamber system over I and $c\in\mathscr{C}$. With $\Delta_J(c)$, $J\subseteq I$, we denote the element in $\mathfrak{P}_J$ containing $c(\Delta_J(c)$ is a chambersystem over J). Let Δ be the diagram of $\mathscr{C}$, $J\subseteq I$, then Δ_J is the diagram which consists of all nodes of Δ not in J and all those bonds which do not contain some element in J. We are going to prove

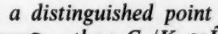
Theorem. *Let $\mathscr{C}$ be a classical, locally finite, nontight Tits chamber system of characteristic 2 over a set I with connected diagram $\Delta=\Delta(I)$ of* rank $\geqq 3$, *and with transitive automorphism group G and finite chamber stabilizer $B=G_c(c\in\mathscr{C})$. For $J\subseteq I$ let G_J be the stabilizer of $\Delta_{I-J}(c)$ and K_J be the kernel of the action of G_J on $\Delta_{I-J}(c)$, $X_i=G_{\Delta_i(c)}$, $i\in I$. Suppose furthermore that Δ contains no bonds of strength >2 but at least one bond is of strength 2. Finally if $J\subseteq I$ such that $\Delta_{I-J}(c)$ belongs to*  *or*  *we assume that $G_J/K_J\ncong A_7$. Then one of the following holds.*

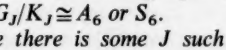
(1) *$\mathscr{C}$ is the chamber system of a finite spherical building and G is an extension of a group of Lie-type Δ by diagonal and field automorphisms.*

(2) *Δ is a complete bipartite graph containing no single bonds and one of the following holds*

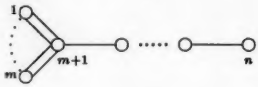
(a) *If $\Delta_{I-J}(c)$ belongs to*  *then $G_J/K_J\cong A_6$ or S_6.*

(b) *If $\Delta_{I-J}(c)$ belongs to*  *then $G_J/K_J\cong\tilde{U}_4(2)$ or $\tilde{U}_5(2)$. Furthermore Δ is a star.*

(c) *There is a distinguished point 1 in Δ such that if $1\in I-J$ and $\Delta_{I-J}(c)$ belongs to* *, then $G_J/K_J\cong\tilde{U}_4(2)$. If $1\notin I-J$ and $\Delta_{I-J}(c)$ belongs to*  *, then $G_J/K_J\cong A_6$ or S_6.*

In any case there is some J such that $\Delta_{I-J}(c)$ belongs to  *($1\in I-J$ in case* (c)*) such that $K_J=1$.*

(3) $|I|=n>3$ *and* Δ *is of type*



$m\geqq 2,\ m+1<n$

For $J=I-K, K\subseteq\{1,\ldots,m\}, |K|=m-1$, *we have* $K_J=1$, $\Delta_{I-J}(c)$ *is a building and* G_J *is an automorphism group of the corresponding group of Lie type.*

(Here and later on we denote by $\tilde{U}_n(2)$ any subgroup of $\mathrm{Aut}(U_n(2))$ containing $U_n(2)$).

It has been shown in [20] that a chamber system with nonconnected diagram is the "direct sum" of chamber systems with connected diagrams. So this assumption in the theorem is not a real restriction.

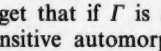
As any parabolic system in the sense of [19] yields a chamber system with the same diagram, we get a similar theorem for parabolic systems, which can be worked out as an exercise.

Suppose now Γ is a geometry over I. Then one can define the chamber system $\mathscr{C}(\Gamma)$ in the following way: The chambers are the flags of Γ of type I, two such chambers being i-adjacent if and only if they coincide in a flag of type $I-\{i\}$. Then $\mathscr{C}(\Gamma)$ is connected if Γ is residually connected. Furthermore from [19] we get that if Γ is a classical Tits geometry over I with diagram Δ and flag-transitive automorphism group G, then $\mathscr{C}(\Gamma)$ is a classical Tits chamber system over I with the same diagram Δ and G is a chamber transitive automorphism group of $\mathscr{C}(\Gamma)$.

If $\mathscr{C}(\Gamma)$ has characteristic p we say that Γ is of characteristic p.

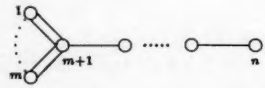
Now we get with our theorem and the results of [3, 12] and [14] the following corollary:

Corollary. *Let* Γ *be a finite classical Tits geometry of characteristic* 2 *over* I, $|I|\geqq 3$, *with connected diagram* $\Delta=\Delta(I)$. *Suppose that* Δ *contains bonds of strength* 2 *but no bonds of higher strength. Suppose that* Γ *admits a flag transitive automorphism group* G. *For* $J\subseteq I$ *denote by* G_J *the stabilizer of the residue* Γ_F *of a flag* F *of type* J *and* K_J *the kernel of the action of* G_J *on* Γ_F. *Then one of the following holds.*

(1) Γ *is a finite building and* G *is an extension of the corresponding simple group of Lie-type by diagonal and field automorphisms or* A_7, *or* $\Delta=$  *and* $G\cong A_7$.

(2) Δ *is a complete bipartite graph, all of whose bonds are of strength* 2 *and* $\mathscr{C}(\Gamma)$ *is as in Theorem* (2).

(3) Δ *is of type*



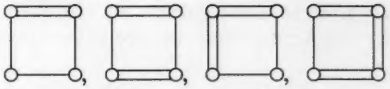
$m\geqq 2,\ n\neq m+1$

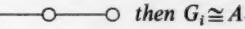
and one of the following holds

(a) $\mathscr{C}(\Gamma)$ *is as in Theorem* (3), *or*

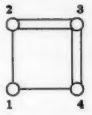
(b) $n=m+2$, $G_{\{2,\ldots,m\}}$ *is an extension of an elementary abelian group of order* 64 *by* A_7 *and* $G_J\cong \mathrm{Sp}_6(2)$ *for any* $J\subseteq\{1,\ldots,m\}$, *with* $|J|=m-1$ *and* $1\in J$.

(4) *Δ is one of the following*



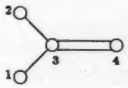
If G_i belongs to *or*  *then $G_i \cong A_7$. In the first two cases $G \cong U_3(5)$.*

(5) *Δ is of type*



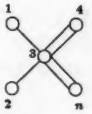
$G_2 \cong G_4$ is an extension of an elementary abelian group of order 16 by A_7 and $G_3 \cong \hat{A}_8$.

(6) *Δ is of type*



$G_1 \cong G_2 \cong \mathrm{Sp}_6(2)$ and G_4 is an extension of an elementary abelian group of order 64 by A_7.

(7) *Δ is of type*

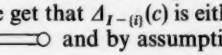


$n \geqq 5$

For any $J \subseteq \{4, \ldots, n\}$, $|J| = n-4$, we have that G_J is as in (6). In particular $K_{J \cup \{1\}} = K_{J \cup \{2\}} = 1$.

Similar results for classical, locally finite Tits chamber systems of characteristic $p > 3$, may be found in [19]. Thus the only remaining case is just $p = 3$.

In fact all the possibilities in the theorems and the corollary really occur. We restrict ourselves to the corollary. Examples of type (2) and (3) have been constructed by Aschbacher, Kantor, Meixner, Smith or Wester. Some of them may be found in Habilitationsschrift of *M.* Wester [21], in [8], or [1, 4]. For $n = 3$ we have interesting examples like $U_4(p)$ [5] and Suz [9]. The last two diagrams in (4) can be realized in $\Omega_6^-(3)$ see [21]. Type (5) can be realized in McLaughlin's simple group [11]. Type (6) can be realized in $\Omega_7(p)$ [1]. Type (7) can be constructed using the ideas of [8].

Now some words about the proof of the theorem. By [13] we may assume that Δ contains  . Now we argue by induction. If $\mathrm{rank}(\Delta) = 3$, then the result follows from [18] (see (1.8)). So we first assume that $\mathrm{rank}(\Delta) = 4$. This gives us a very restricted list of possible diagrams. Let $i \in I$, such that Δ_i is connected then we get that $\Delta_{I-\{i\}}(c)$ is either a building or a chambersystem belonging to and by assumption the latter occurs. Now in most

cases the structure of the groups belonging to the latter chambersystem as described in (1.8) yields that the groups of Lie type (if there are any) are defined over $GF(2)$. This then gives that the kernels K_i are 2-closed, mostly even 2-groups. Now G_i/K_i acts on K_i and there are two cases to consider: $\mathbf{C}_B(K_i) \supsetneq K_i$ and $\mathbf{C}_B(K_i) \subseteq K_i$.

If one checks the groups described in the theorem for $\text{rank}(\Delta)=4$, one realizes that the generic case will be $K_i=1$. This means that $\mathbf{C}_B(K_i) \supsetneq K_i$ will lead to the diagrams and groups as in the theorem. Now in this situation it turns out that just by comparing the structures of different G_j's, it is an easy job to get $K_i=1$.

So assume that $\mathbf{C}_B(K_i) \subseteq K_i$ for the "relevant" G_i's. Let $S \in \text{Syl}_2(B)$, then set $Z_i=\langle \Omega_1(Z(S))^{G_i} \rangle \subseteq K_i$. Now G_i acts on Z_i. We are able to determine the action of G_j on Z_j for different j's. It then turns out that these actions do not fit together, giving a contradiction. As an example for this see (1.12)–(1.14). It is worthwhile mentioning that the G_i's have not to be finite. Nevertheless we are able to prove properties of finite representations for such groups.

After having treated the case $\text{rank}(\Delta)=4$ the remainder of the proof is just an easy exercise in induction. We just have to pin down all diagrams where any connected subdiagram containing a double bond is one of the diagrams described in the theorem. Just by combinatorics and the fact that $\text{rank}(\Delta)>4$ we get that Δ is one of the diagrams described in the theorem.

For the proof of the theorem in Chap. 2 and 3 we fix the notation of the theorem. If $J=\{a, b, c \ldots\}$ we also write $G_{a,b,c\ldots}$ and $K_{a,b,c\ldots}$ instead of G_J or K_J. Furthermore we set $Z_J=\langle \Omega_1(Z(S))^{G_J} \rangle$, where $S \in \text{Syl}_2(B)$.

If G is a finite group V a faithful $GF(2)$ G-module, $A \neq 1$ an elementary abelian 2-subgroup of G with $|V: \mathbf{C}_V(A)| \leqq 2^i |A|$ for some $i \in \mathbb{N} \cup \{o\}$, then V is called an F_i-module with offending subgroup A. An F_0-module is also called an F-module. All other notations will be standard.

§ 1. Preliminaries

For convenience of the reader we summarize some of the preliminary lemmas contained in [13].

(1.1) **Lemma.** *Let G be one of $\text{Sp}_{2n}(q)$, $\Omega^-(2n, q)$, $(S)U_n(q)$, $n \geqq 3$, $q=2^f$, and V an irreducible $GF(2)$ G-module. If V is an F_1-module, then*

(a) *$G \cong \text{Sp}(2n, q)$, V is an F-module and V is the natural module or $n=3$ and V is the spinmodule, or V is not an F-module $n=4$, $q=2$ and V is the 16-dimensional spinmodule*

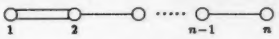
(b) *$G \cong \Omega^-(2n, q)$, V is the natural module,*

(c) *$G \cong (S)U_n(q)$, V is the natural module, or $n=4$ and V is the natural module for $\Omega_6^-(q)$ $(\cong U_4(q))$.*

Proof. [2; (2.1)], [6; (6.40)].

(1.2) **Lemma.** *Let $G=\text{Sp}(2n, q)$, $q=2^f$, V an irreducible $GF(q)$-module, $G=\langle X_1, \ldots, X_n \rangle$, where the X_i's are the minimal parabolics containing a fixed*

Borel subgroup corresponding to the following diagram



Let $S \in \mathrm{Syl}_2(X_1)$.

(a) *If* V *is the natural module, then*

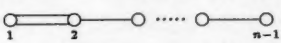
$$N_G(\mathbb{C}_V(S)) = \langle X_1, \ldots, X_{n-1} \rangle$$

(b) *If* V *is the spinmodule, then* $N_G(\mathbb{C}_V(S)) = \langle X_2, \ldots, X_n \rangle$.

If $q = 2$ *the converse of* (a) *and* (b) *is true.*

Proof. Well known [10].

(1.3) **Lemma.** *Let* $G \cong \Omega^-(2n, q)$, $(S)\, U(2(n-1), q)$ *or* $(S)\, U(2n-1, q)$, $q = 2^f$, $G = \langle X_1, \ldots, X_{n-1} \rangle$, *where the* X_i*'s are the minimal parabolics containing a fixed Borel subgroup corresponding to the following diagram*



For $n = 3$ *choose notation such that* $O^{2'}(X_1/O_2(X_1)) \cong L_2(q^2)$ *for* $\Omega^-(6, q)$ *and* $O^{2'}(X_2/O_2(X_2)) \cong L_2(q^2)$ *otherwise. Let* V *be an irreducible* $GF(2)$*-module for* G *and* $S \in \mathrm{Syl}_2(X_1)$. *Then*

(a) *If* V *is the natural module, then* $N_G(\mathbb{C}_V(S)) = \langle X_1, \ldots, X_{n-2} \rangle$.

(b) *If* $G = U_4(q)$ *and* V *is the natural* $\Omega_6^-(q)$*-module, then* $N_G(\mathbb{C}_V(S)) = X_2$.

(1.4) **Lemma.** *Let* G *be* $\Omega^-(2n, q)$, $q = 2^f$, $n > 3$. *Denote by* $\tilde{G}$ *the stabilizer of a singular point in the natural representation of* G. *Let* V *be a nontrivial* $GF(2)$ G*-module and* $1 \neq A \subseteq O_2(\tilde{G})$. *If* V *is an* F*-module, then* A *is not an offending subgroup. If* V *is an* F_1*-module and* A *is an offending subgroup, then* $q = 2$.

Proof. An easy calculation shows $|V\colon \mathbb{C}_V(A)| \geqq q\,|A|$ for any $1 \neq A \subseteq O_2(\tilde{G})$.

(1.5) **Lemma.** *Let* $G = \langle G_1, G_2 \rangle$, $K_i \trianglelefteq G_i$, $|K_i| < \infty$, $i = 1, 2$, *and* $K_1 \subseteq G_2$, $K_2 \subseteq G_1$. *Suppose there is no nontrivial normal subgroup* N *of* G *contained in* $G_1 \cap G_2$. *Let* p *be a prime with* $p \nmid |K_1 K_2/K_1|$, $p \nmid |K_2 K_1/K_2|$, *then* $p \nmid |K_1 K_2|$.

Proof. [13; (1.12)].

(1.6) **Lemma.** *Let* $\mathscr{C}$ *be a connected chamber system over* $I = \{1, \ldots, n\}$, $n \geqq 2$, *with chamber transitive automorphism group* G. *Fix a chamber* c *and set* $B = G_c$. *Assuming that* B *is finite. Set* $X_i = G_{\Delta_i(c)}$ *and let* Q_i *be the kernel of the action of* X_i *on* $\Delta_i(c)$. *Suppose that* B/Q_i *is a 2-group for* $i = 1, \ldots, n$, *then also* B *is a 2-group.*

Proof. [13; (1.13)].

(1.7) **Lemma.** *Let* $G = \langle G_1, G_2 \rangle$ *with* $|G_2| < \infty$ *and*

(i) $F^*(G_2) = O_2(G_2)$.

(ii) $G_1 \cap G_2 = B = N_{G_2}(S)$, $S \in \mathrm{Syl}_2(G_1 \cap G_2)$.

(iii) *No nonidentity subgroup of* S *is normal in both* G_1 *and* G_2.

(iv) $O^{2'}(G_2/O_2(G_2)) \cong S_3$.

(v) $Z_1 \subseteq \mathbb{C}_S(Z_1) \trianglelefteq G_1$, *where* $Z_i = \langle \Omega_1(Z(S))^{G_i} \rangle$, $i = 1, 2$.

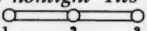
Then either

(1) Z_1 *is an* F_1*-module for* $G_1/\mathbb{C}_{G_1}(Z_1)$ *with offending subgroup in* $S/\mathbb{C}_S(Z_1)$,

or

(2) $Z_2 \neq \Omega_1(Z(S))$, $Z_1 Z_2 \lhd G_2$ *and if* $\rho \in O^{2'}(G_2)$, $o(\rho)=3$, *then* $|[\rho, O_2(G_2)]|=4$.

Proof (cf. [16; (2.1), (2.2)]). (Unfortunately, [loc.cit.] also assumes $F^*(G_1)=O_2(G_1)$ but what is really used is that $\mathbf{C}_S(Z_1) \lhd G_1$, which we assume to be true by (v).)

(1.8) **Lemma.** *Let $\mathscr{C}$ be a classical locally finite nontight Tits chamber system of characteristic two with diagram Δ of type* 

, and transitive automorphism group G with finite chamber stabilizer $B=G_C$. For $i \in I=\{1, 2, 3\}$ let G_i be the stabilizer of $\Delta_{I-\{i\}}(c)$ and K_i the kernel of the action of G_i on $\Delta_{I-\{i\}}(c)$ and $Q_i=O_2(K_i)$. Then one of the following holds:

(i) $G_i/K_i \cong S_6$ *or* A_6, $i=1, 3$. $\mathbf{C}_{G_i}(K_i) \subseteq K_i$ *for* $i=1, 3$, $2^4 \leqq |K_i| \leqq 2^5$, $Z(G_i)=1$, $i=1, 3$ *and* $\Omega_1(Z(S)) \lhd G_2$, $S \in \mathrm{Syl}_2(B)$.

(ii) $G_1 \cong G_3 \cong S_6$ *or* A_6 *and* $K_1=K_3=1$, $G_2/K_2 \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \cdot 2$.

(iii) $G_1 \cong G_3 \cong \Omega_6^-(2)$ *or* $O_6^-(2)$, $K_1=K_3=1$, $G_1 \cap G_3$ *involves* A_5.

(iv) $G_1/K_1 \cong \Omega_6^-(2)$ *(or* $O_6^-(2)$*)*, $G_3/K_3 \cong A_6$ *(or* S_6*)*, K_1 *is extraspecial of order* 2^7, K_3 *is 2-closed with* $|K_3 : Q_3|=3$, $G_3/Q_3 \cong 3A_6$ *(or* $3S_6$*)* *and* Q_3 *is special with* $|Z(Q_3)|=2^4$, $|Q_3/\phi(Q_3)|=2^6$. *Furthermore* $Q_3/\phi(Q_3)$ *and* $\phi(Q_3)$ *are irreducible* G_3/Q_3*-modules.* K_2 *is special of order* 2^{10} *with* $|Z(K_2)|=4$ *and* $G_2/K_2 \cong A_5 \times S_3$ *(or* $S_5 \times S_3$*)*.

(v) $G_1^\infty \cong G_3^\infty \cong U_5(2)$, $K_1=K_3=1$, $G_1 \cap G_3$ *involves* A_5.

Proof [18; Theorem 1].

The next lemma is in the spirit of [7].

(1.9) **Lemma.** *Let G be as in* (1.8)(i) *and V be a finite nontrivial irreducible $GF(2)$ G-module, then there is no $W \subseteq K_i$, $i=1$ or 3, $|W|=4$ such that $[V, W, W]=1$.*

Proof (see [7; (3.2)]). Denote by Q_i, $i=1, 3$, the subgroup of K_i, such that $|Q_i|=16$ and $Q_i \lhd G_i$. By Q_2 we denote the extraspecial normal subgroup of G_2 of order 32. We also denote Q_2 by Q_z where $\langle z \rangle = \Omega_1(Z(S))$.

We will show that we may assume $W \subseteq Q_i$. So suppose not. Then $|K_i : Q_i|=2$ and $|W \cap Q_i|=2$. We may assume that $W \cap Q_i = \Omega_1(Z(S))$. Furthermore K_i is an indecomposable module for G_i/K_i and so G_i/K_i possesses orbits of length 6 and 10 on $K_i \setminus Q_i$. Let $w \in W \setminus Q_i$, then in both cases $\mathbf{C}_{G_i}(w)$ acts irreducibly on Q_i. Hence $[[V, w], Q_i]=1$. In particular any foursgroup in K_i containing w acts quadratically. Let $t \in S \setminus \mathbf{C}_S(w)$. Then $[w, t]=x \in Q_i$ and $\mathbf{C}_{Q_i}(t) \neq \langle x \rangle$. Choose $y \in \mathbf{C}_{Q_i}(t) \setminus \langle x \rangle$. Then $\langle y, w \rangle$ acts quadratically and $\langle y, xw \rangle$ acts quadratically. Hence also $\langle y, x \rangle \subseteq Q_i$ acts quadratically. So we assume:

(1) $W \subseteq Q_i$.

Now the action of G_i on Q_i shows that we may assume

(2) $W=Q_1 \cap Q_3$.

As $\mathbf{C}_{G_2}([z, V]) \lhd G_2$, we get $[Q_2, [z, V]]=1$.

(3) $[V, Q_i, Q_i, O^2(G_i)]=1$, $i=1, 3$.

We have $|Q_i : Q_i \cap Q_2|=2$. Choose $g \in G_i$ such that $Q_i = \langle z^g \rangle (Q_i \cap Q_2)$. Then $\langle Q_z, Q_{zg}, Q_{zzg} \rangle \geqq O^2(G_i)$ and hence $[V, z, Q_i]=[V, z, z^g] \leqq \mathbf{C}_V(O^2(G_i))$.

(4) $[\mathbf{C}_V(Q_i), Q_2, Q_2]=1$, $i=1, 3$.

Let $h \in G_2 \setminus G_i$, then $Q_2=(Q_2 \cap Q_i)(Q_2 \cap Q_i^h)$ and thus $[\mathbf{C}_V(Q_i), Q_2, Q_2] \leqq \mathbf{C}_V(Q_i) \cap [V, Q_i^h, Q_i^h] \leqq \mathbf{C}_V(\langle Q_i, O^2(G_i)^h \rangle)$ by (3). Since $\mathbf{C}_V(G)=1$ and $O^2(G_2) \subseteq \langle Q_i, O^2(G_i)^h \rangle$, we get $G=S\langle Q_i, O^2(G_i)^h \rangle$ and so $[\mathbf{C}_V(Q_i), Q_2, Q_2]=1$.

(5) $[\mathbb{C}_V(Q_i), Q_j, Q_j]=1$ where $\{i,j\}=\{1,3\}$.

This follows from $Q_j=(Q_i\cap Q_j)(Q_i\cap Q_j)^g$ for suitable $g\in G_j$ and $[V, Q_i\cap Q_j, Q_i\cap Q_j]=1$ by (2).

Now choose notation so that $[\mathbb{C}_V(S), O^2(G_1)]\neq 1$. Then on $\mathbb{C}_V(Q_1)$ both classes of fours-groups in $O^2(G_1)/Q_1\cong A_6$ operate quadratically by (4) and (5). But A_6 does not possess such a module. This contradiction proves the lemma.

(1.10) **Lemma.** *Let G be as in* (1.8)(i) *and V be a finite irreducible nontrivial $GF(2)$ G-module. Set $L_i=O^2(G_i)$ and $Q_i=O_2(L_i)$, $i=1,3$. Furthermore set $Q_2=K_2\cap L_1$. Choose notation such that $[\mathbb{C}_V(Q_1), L_1]\neq 1$. Then the following holds:*

(1) $|V\colon \mathbb{C}_V(Q_1)|\geqq 2^5$ *and L_1 acts nontrivially on $V/\mathbb{C}_V(Q_1)$.*

(2) *If $t\in Q_i^\#$, $i=1,3$ then $|[V,t]|\geqq 16$.*

(3) *If $K_i>Q_i$, for $i=1$ or 3, then $|[V,t]|\geqq 64$ for $t\in Q_1^\#$. Furthermore $|[V,t]|\geqq 2^5$ for $t\in K_i\setminus Q_i$.*

(4) *If $s\in G_1\setminus K_1$, then $|[V,s]|\geqq 8$.*

(5) *There is no elementary abelian subgroup $A\neq 1$ of S such that $|V\colon \mathbb{C}_V(A)|\leqq 2|A|$ (i.e. G possesses no F_1-module with offending subgroup contained in S.)*

Proof. As $[V, Q_1, Q_1]\neq 1$ by (1.9) we have $[Q_1, V/\mathbb{C}_V(Q_1)]\neq 1$. This shows $|V\colon \mathbb{C}_V(Q_1)|\geqq 2^5$ which is (1).

Furthermore $A_6\cong L_1/Q_1$ induces a nontrivial composition factor in $V/\mathbb{C}_V(Q_1)$ and $\mathbb{C}_V(Q_1)$ and so $|V\colon \mathbb{C}_V(t)|\geqq 16$ for any $t\in L_1\setminus Q_1$. But all involutions in $Q_1\cup Q_3$ are conjugate to some involution in $L_1\setminus Q_1$ which gives (2).

Suppose next $K_i>Q_i$ for some $i=1$ or 3. Let first $t\in K_i\setminus Q_i$, then $\mathbb{C}_{L_i}(t)\cong E_{16}A_5$ or $E_{16}(\mathbb{Z}_3\times\mathbb{Z}_3)\mathbb{Z}_4$, where $O_2(\mathbb{C}_{L_i}(t))$ is an irreducible $\mathbb{C}_{L_i}(t)$-module. If $|[V,t]|\leqq 16$, then the structure of $GL(4,2)$ shows that $[O_2(\mathbb{C}_{L_i}(t)), [V,t]]=1$. This implies that any $W\subseteq K_i$, $|W|=4$, containing t, acts quadratically on V. This contradicts (1.9). So we have $|V\colon \mathbb{C}_V(t)|\geqq 2^5$.

Let now $t\in Q_1$. We may assume that $t\in Z(G_2)$. As $K_i>Q_i$, there is $W\subseteq K_i$, $|W|=4$, $t\in W$ and $W\trianglelefteq G_2$. Now W acts on $[V,t]$. As there is no involution in $GL(5,2)$ centralized by an elementary abelian group of order 9, we get $[[V,t], W]=1$ if $|[V,t]|\leqq 32$, contradicting (1.9). This proves (3).

For proving (4) choose $s\in G_1\setminus L_1$ with $|[V,s]|\leqq 4$. There is some $t\in Q_1^\#$ with $s\sim st$. Hence $|[V,t]|\leqq 16$ and so by (3) $Q_1=K_1$. Now either $|\mathbb{C}_{Q_1}(s)|=8$ or $|\mathbb{C}_{Q_1}(s)|=4$ and there is some $\rho\in G_1$, $o(\rho)=3$, $[\rho,s]=1$ and $[\mathbb{C}_{Q_1}(s),\rho]=\mathbb{C}_{Q_1}(s)$. As $|[V,s]|\leqq 4$ we get in both cases a subgroup W of order 4 in $\mathbb{C}_{Q_1}(s)$ such that $[W,[V,s]]=1$. Hence $\langle w,s\rangle$ acts quadratically on V for any $w\in W^\#$. Now choose $g\in Q_1$ with $s^g=st\neq s$. Then there is some $w\in W^\#$, $w\neq t$. Hence $\langle w, st\rangle$ and $\langle w,s\rangle$ act quadratically, but then also $\langle w,t\rangle$ acts quadratically, contradicting (1.9).

Assume (5) is false. Choose $A\subseteq S$, with $|V\colon \mathbb{C}_V(A)|\,|\leqq 2|A|$. By (2) and (4) $|A|\geqq 8$. We provide a contradiction in a series of steps.

Suppose $A\cap Q_1=1$. Then $|A|=8$ and $A\cap K_1=1$ by (3). Set $W=A\cap L_1$. Then $|W|=4$ and by (2) $\mathbb{C}_V(A)=\mathbb{C}_V(a)$ for any $a\in W^\#$. Hence W acts quadratically on V. There is some $g\in L_3$ such that $t\in W^g$, $t\in Z(S)$. Hence $W^g\subseteq L_3\cap G_2$. Now $W^g\subseteq\mathbb{C}_{G_2}([V,t])\trianglelefteq G_2$. This shows $Q_2\subseteq\mathbb{C}_{G_2}([V,t])$. But then for $s\in Q_1\setminus\langle t\rangle$, $s\in Q_2$, we have that $\langle s,t\rangle$ acts quadratically on V, contradicting (1.9). So we have shown

(a) $A\cap Q_1\neq 1$.

Suppose $A\trianglelefteq Q_1$. If $A\subseteq K_1$, then by (3) $A=K_1$, and $\mathbb{C}_V(t)=\mathbb{C}_V(A)$ for any $t\in Q_1^\#$. But then Q_1 acts quadratically on V, contradicting (1.9). So we have

$A \not\subseteq K_1$. By the choice of L_1, we have $[A, \mathbf{C}_V(A \cap Q_1)] \neq 1$. Furthermore by (2) $|V: \mathbf{C}_V(A \cap Q_1)| \geqq 16$. This shows $|A| \geqq 16$. If $K_i \neq Q_i$ for some i, then by (3) we even get $|V: \mathbf{C}_V(A \cap Q_1)| \geqq 64$ and so $|A| \geqq 64$. But there are no elementary abelian subgroups of order 64 in S. So we have

(b) *Either* $A \subseteq Q_1$ *or* $Q_i = K_i$ *and* $|A| \geqq 16$.

Suppose again $A \not\subseteq Q_1$. If $|V: \mathbf{C}_V(A \cap Q_1)| \geqq 32$, then we get $|A| = 32$ and so $|A \cap Q_1| = 4$. Now by the choice of L_1 we have that $|\mathbf{C}_V(Q_1): \mathbf{C}_V(Q_1) \cap \mathbf{C}_V(A)| \geqq 4$. Then $|\mathbf{C}_V(A \cap Q_1): \mathbf{C}_V(A)| \geqq 4$ and so $|V: \mathbf{C}_V(A)| \geqq 2^7$, a contradiction. Thus we have $|V: \mathbf{C}_V(A \cap Q_1)| = 16$. Now $|A \cap Q_1| = 2$ and so again $|A: A \cap Q_1| = 8$. Hence again $|\mathbf{C}_V(A \cap Q_1): \mathbf{C}_V(A)| \geqq 4$ which gives $|V: \mathbf{C}_V(A)| \geqq 64 = 4|A|$, a contradiction. So we have

(c) $A \subseteq Q_1$.

By (2) $|A| \geqq 8$. If $|A| = 8$, then A acts quadratically by (2), contradicting (1.9). So we are left with $A = Q_1$. Now $|V: \mathbf{C}_V(Q_1)| = 2^5$. Furthermore by (1.9) $|V: \mathbf{C}_V(t)| = 2^4$ for any $t \in Q_1^\#$. Hence again by (1.9) we have

(d) $\mathbf{C}_V(Q_1) = \mathbf{C}_V(W)$ *for any foursgroup* $W \subseteq Q_1$.

Choose $g \in G_2$ with $|Q_1^g \cap G_1| = 8$. Then $|Q_1^g: Q_1 \cap Q_1^g| = 4$. By (d) $\mathbf{C}_V(Q_1) = \mathbf{C}_V(Q_1^g \cap Q_1) = \mathbf{C}_V(Q_1^g)$. Now $\mathbf{C}_V(Q_1)$ is invariant under $\langle G_1, G_1^g \rangle$. As $G_2 \subseteq \langle G_1, G_1^g \rangle$, we have $\langle G_1, G_1^g \rangle = G$, a contradiction.

For groups of type (1.8)(iv) there is no result like (1.9) as $3 \cdot Suz$ is a group of type (1.8)(iv) possessing quadratic foursgroup. Anyway we prove

(1.11) **Lemma.** *Let G be as in* (1.8)(iv) *and V be a finite nontrivial irreducible $GF(2)$ G-module. Set $L_i = O^2(G_i)$, $i = 1, 2, 3$. Then the following holds*

(1) $[V, Z(G_1), L_1] \neq 1$.

(2) $|[V, Z(G_1)]| \geqq 2^6$.

(3) *There is no elementary abelian subgroup A in S such that* $|V: \mathbf{C}_V(A)| \leqq 4|A|$.

Proof. Let $[[V, Z(G_1)], L_1] = 1$. Then any foursgroup in K_1 which contains $Z(G_1)$ acts quadratically on V. Now choose $g \in G_2 \setminus G_1$ such that $Z(G_1)^g \subseteq K_1$. If $\mathbf{C}_V(Z(G_1)) = \mathbf{C}_V(Z(G_1)^g)$, then $\mathbf{C}_V(Z(G_1))$ is invariant under $\langle L_1, g \rangle S = \langle L_1, L_2 \rangle S = G$, a contradiction. Hence $\mathbf{C}_V(Z(G_1)) \neq \mathbf{C}_V(Z(G_1^g))$ and so as all nontrivial subgroups in $\langle Z(G_1), Z(G_1)^g \rangle$ are conjugate under $N(\langle Z(G_1), Z(G_1)^g \rangle)$ we have $W = [V, Z(G_1)] \cap [V, Z(G_1)^g] \neq 1$. Now W is invariant under $\langle L_1, L_1^g \rangle \supseteq L_2$, a contradiction as before. This proves (1).

As L_1 involves $\Omega_6^-(2)$ (2) is a consequence of (1).

Now assume that there exists an $A \subseteq S$ with $|V: \mathbf{C}_V(A)| \leqq 4|A|$. We provide a contradiction in a series of steps

$$(*) \qquad A \cap K_1 = 1.$$

Suppose false. As all involutions in K_1 are conjugate to the involution in $Z(G_1)$, we get $|V: \mathbf{C}_V(A)| \geqq 2^6$ with (2). In particular $|A| \geqq 16$. Without loss $Z(G_1) \subseteq A$. As $m_2(K_1) = 3$, we have $A \not\subseteq K_1$ and so by (1) and (1.1) $|[V, Z(G_1)]: \mathbf{C}_{[V, Z(G_1)]}(A)| \geqq |A: A \cap Q_1|$. In particular $|\mathbf{C}_V(Z(G_1)): \mathbf{C}_{\mathbf{C}_V(Z(G_1))}(A)| \geqq |A: A \cap Q_1|$. By (2) $|V: \mathbf{C}_V(Z(G_1))| \geqq 2^6$. As $Z(G_1) \subseteq A$ we get $4|A| \geqq |V: \mathbf{C}_V(A)| \geqq 2^6 \cdot |A: A \cap Q_1|$. This now gives $|A \cap Q_1| \geqq 16$, a contradiction.

By (1) and (*) A acts nontrivially on $[V, t]$ and $V/\mathbb{C}_V(t)$, $\langle t\rangle = Z(G_1)$. Hence $|V\colon \mathbb{C}_V(A)| \geqq |A|^2$ and so $|A| \leqq 4$. Now A contains a transvection a on $[V, t]$ since A is offending on $[V, t]$. As $\mathbb{C}_V(t) = [V, t]\,\mathbb{C}_V(A)$ we obtain $|V\colon \mathbb{C}_V(a)| = 4$. (Note that $\mathbb{C}_{G_1/K_1}(aK_1)$ acts irreducibly on $\mathbb{C}_{\bar{V}}(a)/[\bar{V}, a]$, $\bar{V} = V/\mathbb{C}_V(t)$.) As $a \in G_3$, there is some $x \in Z(K_3)$ such that $a \sim ax$ in G_3. Hence $|V\colon \mathbb{C}_V(x)| \leqq 16$. As $\langle x\rangle \sim Z(G_1)$ in G, we have a contradiction to (2).

Next we prove a lemma which is based on the interaction of (1.7), (1.10) and (1.11) and which will be one of the main tools for the proof of the theorem.

(1.12) **Lemma.** *Let $\mathscr{C}$ be a classical locally finite nontight Tits chamber system of characteristic two of rank four with transitive automorphism group G and finite chamber stabilizer $B = G_c$. For $i \in I = \{1, 2, 3, 4\}$ let G_i be the stabilizer of $\Delta_{I-\{i\}}(c)$, K_i be the kernel of G_i on $\Delta_{I-\{i\}}(c)$, and X_i the stabilizer of $\Delta_{\{i\}}(c)$. Denote by S a Sylow 2-subgroup of B. Suppose furthermore*

(i) *G_1/K_1 is as in* (1.8)(i) *or* (1.8)(iv) *with K_1 a 2-group and $\mathbb{C}_S(K_1) \subseteq K_1$.*

(ii) *There is $\tilde{X}_1 \subseteq X_1 \subseteq G_2$ with $\tilde{X}_1/O_2(\tilde{X}_1) \cong S_3$, $S \in \mathrm{Syl}_2(\tilde{X}_1)$, where G_2/K_2 is as in* (1.8)(i), (1.8)(iv) *or $G_2/K_2 \cong \mathrm{Sp}_6(2)$.*

(iii) *$\langle G_1, \tilde{X}_1\rangle = \langle G_1, X_1\rangle = G$ (in fact* (iii) *follows from the other assumptions).*

Then $\Omega_1(Z(S)) \lhd G_1$.

Proof. Suppose false. We are going to show that we have the assumptions of (1.7) for the pair $(G_1, \tilde{X}_1)$.

By assumption B is finite and so S is finite and then by (ii) $\tilde{X}_1$ is finite.

As $\tilde{X}_1 K_2/K_2$ is contained in a parabolic of G_2/K_2 we get from (1.8) resp. the structure of $\mathrm{Sp}_6(2)$ that $F^*(\tilde{X}_1/K_2) = O_2(\tilde{X}_1/K_2)$. So we have (1.7)(i).

By (ii) and (iii) $G_1 \cap \tilde{X}_1 = S = N_{\tilde{X}_1}(S)$ and so we have (1.7)(ii). As $\langle G_1, \tilde{X}_1\rangle = G$ we also have (1.7)(iii). Further (1.7)(iv) follows from the definition of $\tilde{X}_1$.

Set $Z_1 = \langle \Omega_1(Z(S))^{G_1}\rangle$. Then by assumption $Z_1 \neq \Omega_1(Z(S))$. As $\mathbb{C}_S(K_1) \subseteq K_1$, we have $Z_1 \subseteq Z(K_1)$. Since by the structure of G_1/K_1 described in (1.8) there is no normal subgroup N of G_1/K_1 with $N \cap S/K_1 \neq 1$ we have $\mathbb{C}_S(Z_1) = K_1 \lhd G_1$ giving (1.7)(v).

Now we may apply (1.7) and so we get two possibilities. If Z_1 is an F_1-module with offending subgroup in S/K_1 we get a contradiction to (1.10) or (1.11).

So we may assume that for $\rho \in \tilde{X}_1$, $o(\rho) = 3$, we get $|[O_2(\tilde{X}_1), \rho]| = 4$. But now an easy inspection of the groups in (1.8)(i), (1.8)(iv), $\mathrm{Sp}_6(2)$, respectively, shows that $|[O_2(\tilde{X}_1 K_2/K_2), \rho]| \geqq 16$. This proves the lemma.

(1.13) **Lemma.** *Let $\mathscr{C}$ be a classical locally finite Tits chamber system of rank four with transitive automorphism group G and finite chamber stabilizer $B = G_c$. For $i \in \{1, 2, 3, 4\} = I$ let G_i be the stabilizer of $\Delta_{I-\{i\}}(c)$ and K_i be the kernel of G_i on $\Delta_{I-\{i\}}(c)$. Assume further that K_1 and K_2 are 2-groups and the following hold:*

(i) *$G_1/K_1 \cong \mathrm{Sp}_6(2)$ and G_2/K_2 is as in* (1.8)(i).

(ii) *$G_1 \cap G_2/K_1$ is the stabilizer of a point in the natural representation of* $\mathrm{Sp}_6(2)$ (*i.e. $G_1 \cap G_2/K_1 \cong 2\backslash 2^4 \backslash S_6$*).

(iii) *$\Omega_1(Z(S)) \lhd G_2$, where $S \in \mathrm{Syl}_2(B)$, $\mathbb{C}_S(K_1) \subseteq K_1$.*

Then there exists a $\bar{Z}_1 \lhd G_1$, $\bar{Z}_1 \subseteq K_1$ such that $\bar{Z}_1$ is the natural module for $\mathrm{Sp}_6(2) \cong G_1/K_1$.

Proof. The structure of G_2 in (1.8)(i) shows $G_2 = \langle S^{G_2} \rangle$ and so $\Omega_1(Z(S)) \lhd Z(G_2)$. Set $Z_1 = \langle \Omega_1(Z(S))^{G_1} \rangle$. Then $Z_1 \cap Z(G_1) = 1$. Let X_1 be the parabolic in G_2 containing S but not contained in $G_1 \cap G_2$.

Now we are going to apply (1.7) to the pair (G_1, X_1). We have $|X_1| < \infty$ and as $X_1/K_2 \subseteq G_2/K_2$ we see from (1.8)(i) that $F^*(X_1/K_2) = O_2(X_1/K_2)$ and so $F^*(X_1) = O_2(X_1)$. As $X_1/O_2(X_1) \cong S_3$, we have $X_1 \cap G_1 = S \in \mathrm{Syl}_2(G_1 \cap X_1)$. So we have that (1.7)(i)–(iv) are satisfied.

As $\mathbf{C}_{G_1}(Z_1) \lhd G_1$, we get $\mathbf{C}_{G_1}(Z_1) = G_1$ or K_1. But $Z(G_1) = 1$, and so $\mathbf{C}_{G_1}(Z_1) = K_1$.

Now (1.7) implies that either Z_1 is an F_1-module or there is some $\rho \in X_1$, $o(\rho) = 3$, such that $|[O_2(X_1), \rho]| = 4$. But as $X_1 \subseteq G_2$, we see from (1.8)(i) that $|[O_2(X_1/K_1), \rho]| \geqq 16$ for any ρ, and so we have that Z_1 is an F_1-module.

By (1.1) Z_1 contains a submodule $\tilde{Z}_1$ which is either the natural module or the spinmodule. As $G_1 \cap G_2$ centralizes $\Omega_1(Z(S)) \cap \tilde{Z}_1$, we get with (1.2) that $\tilde{Z}_1$ is the natural module, which is the assertion.

For the proof of the next lemma we need the concept of the so-called coset graph. We are going to give definitions and some properties. For more information see [16].

Let H be a group and H_1, H_2 be two different subgroups of H with $H = \langle H_1, H_2 \rangle$. The graph $\Gamma = \Gamma(H_1, H_2)$ has as its vertices all the right cosets of H_1 and H_2 in H, and two cosets are adjacent if they are distinct and have nonempty intersection. The group H acts on Γ by right multiplication. Two vertices δ and λ of Γ are said to be conjugate ($\delta \sim \lambda$) under H if they are in the same H-orbit of Γ. With $d(\delta, \alpha)$ we denote the usual distance metric on Γ and $\Delta(\delta) = \{\alpha \in \Gamma : d(\delta, \alpha) = 1\}$. The following elementary properties can be easily checked:

(i) Γ is bipartite and connected.

(ii) $H \subseteq \mathrm{Aut}(\Gamma)$ iff there is no non-trivial normal subgroup of H contained in $H_1 \cap H_2$.

(iii) H operates edges-transitively on Γ.

(iv) The vertex-stabilizers H_δ, $\delta \in \Gamma$, are conjugate in H to H_1 or H_2.

(v) H_δ is transitive on $\Delta(\delta)$ for $\delta \in \Gamma$.

(vi) The edge-stabilizers are conjugate in H to $H_1 \cap H_2$.

Usually we identify the vertex H_1 with 1 and H_2 with 2. If X_i is subgroup of H_i, $i = 1, 2$, $\alpha \in \Gamma$, $\alpha \sim i$, then we denote by X_α the corresponding subgroup of H_α.

(1.14) **Lemma.** *Let the assumptions be as in* (1.13). *Then* $\mathbf{C}_S(K_2) \supsetneq K_2$.

Proof. We choose notation as in (1.13). Then there is a subgroup $\tilde{Z}_1 \lhd G_1$, such that $\tilde{Z}_1$ is the natural module. Assume $\mathbf{C}_S(K_2) \subseteq K_2$. We have $G = \langle G_1, G_2 \rangle$. Let X_2 be the parabolic in G_1 with $X_2 \supsetneq G_1 \cap G_2$. Then $G = \langle X_2, G_2 \rangle$. To simplify notation set $H_1 = X_2$, $H_2 = G_2$, $Q_1 = O_2(X_2)$, $Q_2 = K_2$. To prove the lemma, we use the coset graph $\Gamma(H_1, H_2)$.

Let $z \in \Omega_1(Z(S))^* \cap \tilde{Z}_1$. Then as $\Omega_1(Z(S)) \lhd H_2$, we get $z \in Z(H_2)$. Set $T_1 = \langle z^{X_2} \rangle$, then $|T_1| = 4$ and $T_1 \lhd H_1$. Set $T_2 = \langle z \rangle$.

Let $b = \min\{d(1, \alpha) | \alpha \in \Gamma, T_1 \subseteq H_\alpha, T_1 \supsetneq H_\beta \text{ for some } \beta \in \Delta(\alpha)\}$.

For the remainder of the proof let $(1, \alpha)$ be a pair, with $d(1, \alpha) = b$, $T_1 \subseteq H_\alpha$, $T_1 \supsetneq H_\beta$ for some $\beta \in \Delta(\alpha)$.

(1) *Suppose* $U \subseteq Z(H_2) \cap Q_2$, *or* $U = \phi(Q_2)$. *Then* $\mathbf{C}_{H_1 \cap H_2}(Q_2/U) \subseteq Q_2$.

For any parabolic P of H_2/Q_2 we have that $\mathbf{C}_{H_2}(Q_2/U)\, Q_2/Q_2 \cap P \lhd P$. So we see with (1.8)(i) that $O^2(P) \leqq \mathbf{C}_{H_2}(Q_2/U)\, Q_2/Q_2 \cap P$ for any P or the intersection is always trivial. In the latter we have the assertion. So assume that $O^2(P)$ centralizes Q_2/U, then $O^2(P)$ centralizes Q_2 and so $\mathbf{C}_S(Q_2) \supsetneqq Q_2$, a contradiction.

Set $V_2 = \langle T_1^{H_2} \rangle$.

(2) *If* $V_2 \subseteq Q_2$ *then* $[Q_2, V_2] \subseteq T_2$ *and* $\mathbf{C}_{H_1 \cap H_2}(V_2/T_2) = Q_2$.

We have $T = H_1 \cap H_2 \in \mathrm{Syl}_2(H_1)$ and $[T, T_1] = T_2$. Thus $[Q_2, T_1] \subseteq T_2$ and so $[Q_2, V_2] \subseteq T_2$. As for any parabolic subgroup P of H_2/Q_2 we have that $\mathbf{C}_{H_2}(V_2/T_2)/Q_2 \cap P \lhd P$, we see with (1.8)(i) that $\mathbf{C}_{H_2}(V_2/T_2)/Q_2 \cap P = 1$ and so $\mathbf{C}_{H_1 \cap H_2}(V_2/T_2) = Q_2$.

(3) $b \geqq 2$.

By (2) $|[T_1, Q_2]| \leqq 2$. Application of (1.10) (5) gives $[T_1, Q_2/\phi(Q_2)] = 1$ and so by (1) $T_1 \subseteq Q_2$, i.e. $b \geqq 2$.

(4) $b = 2$ *in particular* $V_2 \subseteq Q_2$, $V_2 \supsetneqq Q_1$.

Assume $b \geqq 3$. Then V_2 is abelian and $V_2 \subseteq Q_2$.

We claim $\alpha \sim 2$. So suppose $\alpha \sim 1$. Let $(1, 2, 3, \ldots, \alpha-3, \alpha-2, \alpha-1, \alpha)$ be a path from 1 to α of length b. Let $O \in \Delta(1) - \{2\}$, then $V_0 \subseteq H_{\alpha-2}$. As $T_{\alpha-2} = T_{\alpha-3} T_{\alpha-1}$ and $T_{\alpha-1} \subseteq T_1$, since $\alpha \sim 1$ and so $T_\alpha \subseteq H_1$, we get $[V_0, T_{\alpha-1}] = [V_0, T_{\alpha-2}] = 1$. This shows $V_0 \subseteq H_{\alpha-1} \cap H_{\alpha-2}$. Now

$$[V_{\alpha-1} \cap H_0, V_0, V_{\alpha-1} \cap H_0] \subseteq [V_{\alpha-1} \cap H_0, V_{\alpha-1} \cap H_0] = 1.$$

Hence $V_{\alpha-1} \cap H_0 \cap H_1/K_0$ acts quadratically on V_0 and so by (1.9) $|V_{\alpha-1} \cap H_0 : V_{\alpha-1} \cap Q_0| \leqq 2$. As $H_1/Q_1 \cong S_3$, we get $|V_{\alpha-1} : V_{\alpha-1} \cap Q_0| \leqq 4$.

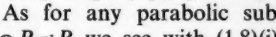
If $[V_{\alpha-1} \cap Q_0, V_0] \neq 1$, then $T_0 = [V_{\alpha-1} \cap Q_0, V_0] \subseteq [V_{\alpha-1}, V_0] \subseteq V_{\alpha-1}$ and so $[V_{\alpha-1}, T_0\, T_2] = 1$, i.e. $[T_\alpha, T_1] = 1$, a contradiction. Thus $|V_{\alpha-1} : \mathbf{C}_{V_{\alpha-1}}(t)| \leqq 4$ for any $t \in V_0^\#$. By (1.10). $V_0 \subseteq Q_{\alpha-1} \subseteq H_\alpha$. As $H_\alpha/\mathbf{C}_{H_\alpha}(T_\alpha) \cong S_3$ we get $[T_\alpha, V_0] = [T_\alpha, T_1] = T_2 \subseteq V_0$. Hence $V_0 \lhd \langle T_\alpha, H_0 \rangle = \langle H_1, H_0 \rangle = G$, a contradiction. This proves the claim.

Now $\alpha \sim 2$ and so $V_2 \subseteq H_\alpha$, $V_\alpha \subseteq H_2$. Thus $[V_2, V_\alpha] \subseteq V_2 \cap V_\alpha$. Again (1.9) gives $|V_\alpha : V_\alpha \cap Q_2| \leqq 2$ and so $|V_\alpha : \mathbf{C}_{V_\alpha}(T_1)| \leqq 4$. As $T_1 \supsetneqq Q_\alpha$, this contradicts (1.10) and $\mathbf{C}_{G_\alpha \cap G_{\alpha-1}}(V_\alpha/T_\alpha) = Q_\alpha$ by (2). So (4) holds.

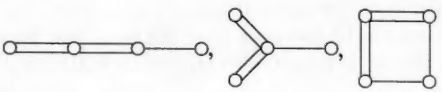
Up to now we never used the action of G_1 on $\tilde{Z}_1$. Suppose $\tilde{Z}_1 \subseteq Q_2$. By (4) $V_2 \subseteq Q_2$ and by (2) $[V_2, \tilde{Z}_1] \subseteq T_2$. In particular $V_2 K_1/K_1$ induces transvections on $\tilde{Z}_1$. As $V_2 K_1/K_1 \subseteq G_1 \cap G_2/K_1$, we have that $V_2 K_1/K_1 \subseteq Z(G_1 \cap G_2/K_1)$. But then, as H_1/K_1 is a parabolic of $\mathrm{Sp}_6(2) \cong G_1/K_1$ containing a Sylow 2-subgroup of $G_1 \cap G_2/K_1$, we have $V_2 \subseteq Q_1$, a contradiction to (4).

So $\tilde{Z}_1 \supsetneqq Q_2$. By (1.8)(i) we have $|\tilde{Z}_1 : \tilde{Z}_1 \cap Q_2| = 16$ or 32 and so $|\tilde{Z}_1 \cap Q_2| = 4$ or 2. But the action of $\mathrm{Sp}_6(2)$ on the natural module implies that $G_1 \cap G_2/K_1$ does not normalize a group of order 4 in $\tilde{Z}_1$ and so $|\tilde{Z}_1 \cap Q_2| = 2$, $|\tilde{Z}_1 : \tilde{Z}_1 \cap Q_2| = 32$. Hence $\tilde{Z}_1 \cap Q_2 = T_2$ and $[Q_2, \tilde{Z}_1] \subseteq Q_2 \cap \tilde{Z}_1 = T_2$. As $T_2 \subseteq Z(G_2)$, we get again the contradiction to (1).

§ 2. Diagrams containing just two double bounds

The aim of this chapter is to treat the case that we have a chamber system $\mathscr{C}$ whose diagram Δ contains  and all other bonds are single bonds. For this chapter we assume that $\mathscr{C}$ fulfills the hypotheses of the theorem.

We first treat the case that $\mathscr{C}$ is of rank four. Then the possible diagrams are:



(Here we used a result of [18] saying that there are no triangle diagrams but)

(2.1) **Lemma.** *Δ is not of type*

Proof. Suppose false. We will get a contradiction in a series of steps. By (1.8) we know the possible local structures for G_4/K_4 and so for $G_{1,4}/K_{1,4}$.

(2.1.1) *G_4/K_4 does not satisfy* (1.8)(v).

Suppose false. Then $G_{1,4}/K_{1,4}$ involves an automorphism group of $U_5(2)$ and so G_1/K_1 involves an automorphism group of $U_7(2)$ which implies that $G_{1,2,4}/K_{1,2,4}$ involves $L_2(4)$, contradicting (1.8)(v).

(2.1.2) *G_4/K_4 does not satisfy* (1.8)(iii).

Suppose false. Then we get $G_{1,4}/K_{1,4} \cong \Omega_6^-(2)$ or $O_6^-(2)$ and so $G_1/K_1 \cong \Omega_8^-(2)$ or $O_8^-(2)$. As K_1K_4/K_4 and K_4K_1/K_1 are both 2-groups (1.5) implies that K_1 and K_4 are both 2-groups. Furthermore by (1.8) $K_1 \subseteq K_4$.

Suppose $\mathbf{C}_S(K_1) \subseteq K_1$. Then $Z(K_4) \subseteq Z(K_1)$. Furthermore $\Omega_1(Z(S)) \neq \Omega_1(Z(K_1))$, since otherwise $\Omega_1(Z(K_1)) = \Omega_1(Z(S)) = \Omega_1(Z(K_4)) \lhd \langle G_1, G_4 \rangle = G$, a contradiction.

If $J(S) \not\subseteq K_1$, then $\Omega_1(Z(K_1))$ is an F-module for G_1/K_1 and thus has to be the natural module by (1.1). Now by (1.4) $J(K_4) = J(K_1) \lhd \langle G_1, G_4 \rangle = G$, again a contradiction. So we get $J(S) \subseteq K_1 \subseteq K_4$ and so $J(K_4) = J(S) = J(K_1) \lhd \langle G_1, G_4 \rangle = G$, the same contradiction as before.

Thus $\mathbf{C}_S(K_1) \not\subseteq K_1$ and so $G_1 = K_1 \mathbf{C}_{G_1}(K_1)$. We have $G_2/K_2 \cong L_3(2) \times S_3$. Choose $t \in G_2 \setminus G_1$. Then t normalizes $O^2(G_1 \cap G_2)K_2$ and so t normalizes $O^2(G_1 \cap G_2)$. Now $(K_1 \cap K_2)^t K_1/K_1$ is centralized by $O^2(G_1 \cap G_2)^t = O^2(G_1 \cap G_2)$. But $G_1 \cap G_2/K_1$ is the stabilizer of an isotropic 3-space in the natural representation of $\Omega_8^-(2)$ and so $O^2(G_1 \cap G_2)$ centralizes no 2-element in $O_2(G_1 \cap G_2/K_1)$. Thus $(K_1 \cap K_2)^t \subseteq K_1$ and so $(K_1 \cap K_2)^t = K_1 \cap K_2$. Then $K_1 \cap K_2 \lhd \langle t, G_1 \rangle = \langle G_2, G_1 \rangle = G$, yielding $K_1 \cap K_2 = 1$. As $|K_1K_2/K_2| = 2$ we get $|K_1| = 2$. Let $\rho \in X_1$, $o(\rho) = 3$ such that $\langle \rho \rangle$ is normalized by K_1. This is possible as $K_1 \not\subseteq K_2$. Now $[K_1, K_2] \subseteq K_1 \cap K_2 = 1$ and so $[\rho, K_2] = 1$. But in G_2 we see that $\langle \rho \rangle K_2/K_2 \lhd G_2/K_2$ and so $\langle \rho \rangle$ is normalized by S. From the structure of G_4 we see that $X_1/K_4 \cong 2^{1+4}(S_3 \times \mathbb{Z}_3)$ or $2^{1+4}(S_3 \times S_3)$. We know that K_4 is a 2-group and so any 3-element of X_1 is contained in $X_1 \setminus K_4$. But in X_1/K_4 there exists no normal subgroup of order 3.

(2.1.3) *G_4/K_4 does not satisfy* (1.8)(iv).

Suppose false. We first show that $G_{1,4}$ involves $\Omega_6^-(2)$. Otherwise we get that $G_1/K_1 \cong \mathrm{Sp}_6(2)$. By (1.5) we have that K_1 and K_4 are $\{2,3\}$-groups. As $O_3(K_1) \subseteq K_4$ and $O_3(K_4) \subseteq K_1$, we get $O_3(K_1) = O_3(K_4) = 1$. Set $Q_i = O_2(K_i)$, $i = 1, 4$. We have $Q_4 \trianglerighteq K_1$. Set $\hat{G}_1 = \langle Q_4^{G_1} \rangle$. Then $[K_1, \hat{G}_1] \leqq Q_1$ as $[K_1, Q_4] \leqq Q_4 \cap K_1 \leqq Q_1$. Hence for $P \in \mathrm{Syl}_3(K_1)$ we get $G_1 = K_1 \mathbf{C}_{G_1}(P)$. But $G_1 \cap G_4$ involves $3 \cdot A_6$ and so $\mathbf{C}_{G_1}(P)$ has to involve a perfect central extension of $\mathrm{Sp}_6(2)$ by a cyclic group of order 3. Such a group does not exist.

So we have that $G_{1,4}/K_{1,4} \cong \Omega_6^-(2)$ or $O^-(6,2)$ and so $G_1/K_1 \cong \tilde{U}_6(2)$. Now G_2/K_2 is involved in $\Sigma_3 \times \mathrm{Aut}(L_3(4))$ and so both K_1K_2/K_1 and K_2K_1/K_2 are 2-groups. Thus (1.5) yields that K_1 and K_2 are 2-groups. Furthermore we see in G_1 that $|K_4 : O_2(K_4)| \leqq 3$ (there might be some diagonal automorphism of order 3 involved in G_1).

Suppose that $\mathbf{C}_S(K_i) \subseteq K_i$ for $i = 1, 4$. If $Z_4 \neq \Omega_1(Z(S))$, then $[X_4, \Omega_1(Z(S))] \neq 1$. Now by [16; (2.2)] we have that Z_4 is an F_2-module for G_4/K_4 with offending subgroup in S/K_4, contradicting (1.11). Hence $Z_4 = \Omega_1(Z(S))$ and so $Z_1 \neq \Omega_1(Z(S))$. Now we apply (1.7) to $(G_1, \tilde{X}_1)$ where $\tilde{X}_1 \subseteq X_1$ with $S \subseteq \tilde{X}_1$ and $\tilde{X}_1/O_2(\tilde{X}_1) \cong S_3$. It is easy to check that $(G_1, \tilde{X}_1)$ satisfies the assumption of (1.7). As $\tilde{X}_1/K_4 \subseteq G_4/K_4$, we have $|[\rho, O_2(\tilde{X}_1/K_4)]| \geqq 16$ for $\rho \in \tilde{X}_1$, $o(\rho) = 3$. Now by (1.7) Z_1 is an F_1-module for G_1/K_1 and so by (1.1) $Z_1/Z_1 \cap Z(G_1)$ is the natural module. But then $SU_6(2)$ is involved in G_1, contradicting $G_1/K_1 \lesssim \mathrm{aut}(U_6(2))$ and $3 \nmid |K_1|$.

So we have that $\mathbf{C}_S(K_1) \trianglerighteq K_1$ or $\mathbf{C}_S(K_4) \trianglerighteq K_4$. We have that $O^{2'}(G_1 \cap G_4/K_1)$ is an extension of an extraspecial group of order 2^9 by an automorphism group of $U_4(2)$ and $O^{2'}(G_1 \cap G_4/K_4)$ is an extension of an extraspecial group of order 2^7 by the same automorphism group. As $|S : K_1(K_4 \cap S)|$ has to be the same viewed in G_1 and G_4 we see that $K_1K_4 = O_2(G_1 \cap G_4)$, a contradiction to $[O^2(G_1 \cap G_4), K_i] = 1$ for $i = 1$ or 4.

Now we can finish the proof of (2.1). We know that G_4/K_4 satisfies (1.8)(i) or (ii) and so $G_1/K_1 \cong \mathrm{Sp}_6(2)$. This gives again that K_1 and K_4 are 2-groups.

(2.1.4) *G_4/K_4 does not satisfy* (1.8)(i).

Suppose false. Let $\mathbf{C}_S(K_i) \subseteq K_i$ for $i = 1, 2$. Application of (1.12) gives $Z_4 = \Omega_1(Z(S))$. Now we have a contradiction with (1.14).

So we may assume that either $\mathbf{C}_S(K_1) \trianglerighteq K_1$ or $\mathbf{C}_S(K_4) \trianglerighteq K_4$. In the latter we get $|K_4K_1 : K_1| \leqq 2$, as $O^2(G_1 \cap G_4)$ centralizes K_4K_1/K_1 and $G_1 \cap G_4/K_1$ is the stabilizer of a point in the natural representation of $\mathrm{Sp}_6(2)$. Now $|S : K_4K_1| \geqq 2^8$. As $|S : K_4| \leqq 2^9$ we get $|K_1 : K_1 \cap K_4| \leqq 2$ and so $[O^2(G_1 \cap G_4), K_1] \subseteq K_1 \cap K_4$. This shows $\mathbf{C}_S(K_1) \trianglerighteq K_1$ by the 3-subgroup lemma.

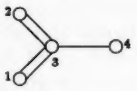
Hence in any case $\mathbf{C}_S(K_1) \trianglerighteq K_1$. As $Z(G_1 \cap G_4/K_4) = 1$, we have $K_1 \subseteq K_4$. But then $[O^2(G_1 \cap G_4), K_4] \subseteq K_1$, as $|K_4 : K_1| \leqq 2$. This shows $G_4 = S\langle O^2(G_1 \cap G_4), O^2(G_3 \cap G_4) \rangle = S\mathbf{C}_{G_4}(K_4)$ and so $K_1 \trianglelefteq \langle G_1, G_4 \rangle = G$. Hence $K_1 = 1$. Now by (1.8) $|K_4| = 2$.

Now $S \in \mathrm{Syl}_2(\mathrm{Sp}_6(2))$ and so S possesses exactly one elementary abelian subgroup of order 64 which is $O_2(\langle X_3, X_4 \rangle)$. But then $O_2(\langle X_3, X_4 \rangle) = K_2$, contradicting $G_2/K_2 \cong \Sigma_3 \times L_3(2)$.

To finish the proof we just have to handle the case that G_4/K_4 satisfies (1.8)(ii). In this case $O_2(\langle X_1, X_2 \rangle) = O_2(\langle X_2, X_3 \rangle) = K_4$. Furthermore $K_1 \subseteq K_4$.

Now $G_3/K_3 \cong A_6 \times S_3$, $S_6 \times S_3$ or $(A_6 \times \mathbb{Z}_3)\cdot 2$. In particular $K_3 \subseteq K_4$ and $|K_4 : K_3| = 2$. Choose $\rho \in X_4$, of order 3. Then ρ normalizes $K_3 K_1/K_1$. As $K_4 K_1/K_1$ is abelian and $\langle \rho \rangle$ is normalized by some element in $K_4 \setminus K_3$ we get $[\rho, K_3] \subseteq K_1$. But it is easy to see that in $\mathrm{Sp}_6(2)$ an element of order 3 never centralizes an elementary abelian group of order 16.

(2.2) **Lemma.** *Let Δ be of type*



then $G_1 \cong G_2 \cong \mathrm{Sp}_6(2)$, $\tilde{U}_6(2)$ or $\tilde{U}_7(2)$.

Proof. Again we split the proof in different subcases.

(2.2.1) *If G_1/K_1 contains $U_6(q)$ or $U_7(q)$, then $G_1 \cong G_2 \cong \tilde{U}_6(2)$ or $\tilde{U}_7(2)$:*

As G_4 is one of the groups in (1.8) we have $q = 2$. Now $\langle X_3, X_4 \rangle$ involves $L_3(4)$ and so G_2/K_2 involves $U_6(2)$ or $U_7(2)$, too. By (1.8) $G_{2,4}/K_4 \cong G_{1,4}/K_4$ and so $G_1/K_1 \cong G_2/K_2$ and $K_1, K_2 \subseteq K_4$.

Now the arguments differ a little bit. Let $G_1/K_1 \cong \tilde{U}_6(2) \cong G_2/K_2$. We have that $K_1 K_2/K_1$ is normal in $G_1 \cap G_2/K_1$. As $G_1 \cap G_2$ acts irreducibly on $K_{1,2}/K_1$, we get $|K_1 K_2/K_1| = 2^9$ or $K_1 = K_2$, in which case $K_1 = K_2 = 1$, which is the assertion. In the first case $K_1 K_2/K_1 = O_2(K_4/K_1)$ is elementary abelian by the structure of $G_{1,2}/K_1$ and extraspecial by the structure of $G_{1,4}/K_1$, a contradiction.

So assume $G_1/K_1 \cong \tilde{U}_7(2) \cong G_2/K_2$. Again we may assume that $K_1 \neq K_2$. Now the parabolic in $U_7(2)$ corresponding to $G_1 \cap G_2$ possesses the normal structure $2^9/2^6/SL_3(4)$. This shows that $K_1 K_2/K_1$ contains an elementary abelian subgroup of order 2^9. But K_4/K_1 is extraspecial of width 5 and so cannot contain such a big elementary abelian subgroup.

(2.2.2) *If $G_1/K_1 \ncong \tilde{U}_6(2)$ or $\tilde{U}_7(2)$, then $G_2/K_2 \cong G_1/K_1 \cong \mathrm{Sp}_6(2)$.*

We have that G_1/K_1 involves $\Omega_8^-(q)$ or $\mathrm{Sp}_6(q)$. Suppose that G_1/K_1 involves $\Omega_8^-(q)$. Then by (1.8) $q = 2$ and X_2 involves $L_2(4)$. Now we get that G_4/K_4 is as in (1.8)(iv) and so $G_2 \cap G_4/K_4$ involves $3 \cdot A_6$. So we get $G_2/K_2 \cong \mathrm{Sp}_6(2)$.

Now both $K_1 K_4/K_4$ and $K_4 K_1/K_1$ are 2-groups. Thus by (1.5) K_1 and K_4 are 2-groups. Hence $K_1 K_2/K_1$ is a 2-group and by the structure of $\mathrm{Sp}_6(2)$ also $K_1 K_2/K_2$ is a 2-group. Now again by (1.5) K_2 is a 2-group. But $G_2 \cap G_4$ involves $3 \cdot A_6$, a contradiction.

So both G_1/K_1 and G_2/K_2 involve $\mathrm{Sp}_6(q)$. Then by (1.8) $q = 2$ and so $G_1/K_1 \cong G_2/K_2 \cong \mathrm{Sp}_6(2)$.

(2.2.3) *If $G_1/K_1 \cong G_2/K_2 \cong \mathrm{Sp}_6(2)$ then $G_1 \cong G_2 \cong \mathrm{Sp}_6(2)$.*

By (1.5) K_1 and K_2 are 2-groups. Now B is a 2-group and so all K_i are 2-groups. Furthermore G_4/K_4 is as in (1.8)(i) or (ii).

Suppose that G_4/K_4 satisfies (1.8)(i). Let $\mathbf{C}_S(K_1) \not\subseteq K_1$. As $Z(G_1 \cap G_2/K_2) = 1$, we get $K_1 \subseteq K_2$ and so $K_1 = K_2 = 1$, the assertion.

So we may assume that $\mathbf{C}_S(K_i) \subseteq K_i$ for $i = 1, 2$. As $|Z(G_1 \cap G_4/K_1)| = 2$, we

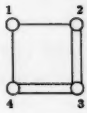
get $\mathbf{C}_S(K_4) \subseteq K_4$ or $|K_4 K_1 : K_1| \leqq 2$. In the latter $K_1 \subseteq K_4$ and so $\mathbf{C}_S(K_1) \supsetneqq K_1$, a contradiction.

So $\mathbf{C}_S(K_i) \subseteq K_i$ for $i = 1, 2, 4$. Application of (1.12) gives $Z_4 = \Omega_1(Z(S))$. Now we get a contradiction with (1.14) applied to the pair (G_1, G_4).

We are left with the case that G_4/K_4 satisfies (1.8)(ii). Then $K_4 = K_{1,4} = K_{2,4}$. Further $K_1 \subseteq K_4 \supseteq K_2$. We have that $K_1 K_2/K_1 \lhd G_1 \cap G_2/K_1$ and so $K_1 K_2/K_1$ is abelian. Suppose $K_2 \nleqq K_1$. Then $K_{1,2} \leqq K_1 K_2 \leqq K_4$, a contradiction to $K_{1,2}/K_1 \nleqq K_{1,4}/K_1$. Thus $K_1 = K_2 = 1$ and we have $G_1 \cong G_2 \cong \mathrm{Sp}_6(2)$, the assertion.

Now (2.2) follows from (2.2.1)–(2.2.3).

(2.3) **Lemma.** *Δ is not of type*



Proof.

(2.3.1) *G_1/K_1 does not satisfy* (1.8)(v).

Suppose false. Then G_2/K_2 involves $U_7(2)$. Hence X_4/K_2 involves $L_2(4)$ while by (1.8) X_3/K_2 involves $L_2(4)$, a contradiction.

(2.3.2) *G_1/K_1 does not satisfy* (1.8)(iv).

Suppose false. Then we may assume that $G_{1,2}/K_{1,2}$ involves $\Omega_6^-(2)$ and so $G_2/K_2 \cong \tilde{U}_6(2)$ since by (1.8) X_4 involves $L_2(4)$. In particular $G_{2,3}/K_{2,3}$ is an automorphism group of $L_3(4)$. This shows that $G_{2,3,4}/K_{2,3,4}$ involves $L_2(4)$. But on the other hand $G_{1,4}/K_{1,4}$ involves A_6 and so $G_4/K_4 \cong \mathrm{Sp}_6(2)$. This shows $G_{2,3,4}/K_{2,3,4} \cong S_3$, a contradiction.

(2.3.3) *G_1/K_1 does not satisfy* (1.8)(iii).

Suppose false. Then G_2/K_2 and G_4/K_4 both involve $\Omega_8^-(2)$. Furthermore K_2, $K_4 \subseteq K_1$ and K_1/K_2, K_1/K_4 are both 2-groups. By (1.5) we get that K_1, K_2 and K_4 are 2-groups.

Suppose $\mathbf{C}_S(K_2) \subseteq K_2$. If $\Omega_1(Z(S)) = \Omega_1(Z(K_2))$, then $\Omega_1(Z(K_1)) = \Omega_1(Z(S))$ and so $\Omega_1(Z(S)) \lhd \langle G_1, G_2 \rangle$, a contradiction. Thus $\Omega_1(Z(S)) \neq \Omega_1(Z(K_2))$. Now $J(K_1) \subseteq K_2$ by (1.4) and so $J(K_1) = J(K_2) \lhd \langle G_1, G_2 \rangle = G$, again a contradiction.

So we have $\mathbf{C}_S(K_i) \supsetneqq K_i$ for $i = 2, 4$. But then $K_2 \cap K_4 \lhd \langle K_i, \mathbf{C}_{G_i}(K_i) | i = 2, 4 \rangle = G$ and thus $K_2 \cap K_4 = 1$. Hence $|K_2 K_4/K_i| \leqq 2$, $i = 2, 4$, since $G_i = K_i \mathbf{C}_{G_i}(K_i)$ we obtain $|K_i| \leqq 2$, $i = 2, 4$. This gives $2^6 \leqq |K_1| \leqq 2^7$. We have $G_3/K_3 \cong A_8$. This shows $K_2 \subseteq K_3$ and so $G_3 \cap G_2/K_3 \cong E_8 L_3(2)$. Then $G_2 \cap G_3/K_2$ possesses a factorgroup $E_8 L_3(2)$, which contradicts [15; (1.12)].

(2.3.4) *G_1/K_1 does not satisfy* (1.8)(i).

Suppose false. Then we get that $G_2/K_2 \cong G_4/K_4 \cong \mathrm{Sp}_6(2)$. Thus by (1.5) K_2, K_4 are 2-groups and so by (1.6) all K_i are 2-groups. Claim $\mathbf{C}_S(K_i) \subseteq K_i$, $i = 1, 2, 4$. Suppose this is false. If $\mathbf{C}_S(K_2) \supsetneqq K_2$, then as $Z(G_2 \cap G_1/K_1) = 1$, we get that $K_2 \subseteq K_1$. Now $|K_1 : K_2| \leqq 2$ and so $\mathbf{C}_S(K_1) \supsetneqq K_1$. The same applies for

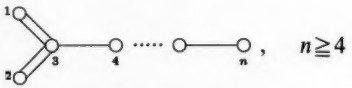
$\mathbf{C}_S(K_4) \trianglerighteq K_4$. So without loss $\mathbf{C}_S(K_1) \trianglerighteq K_1$. Then as $|Z(G_1 \cap G_2/K_2)| = 2$, we get $|K_1 K_2 : K_2| \leqq 2$ and so again $K_2 \subseteq K_1$ and $\mathbf{C}_S(K_2) \trianglerighteq K_2$.

So we may assume $K_2 \subseteq K_1 \supseteq K_4$ and $\mathbf{C}_S(K_1) \trianglerighteq K_1$. Hence $K_2 = K_4 = 1$. Now $|K_3| = 8$ and as $G_3/K_3 \cong A_8$, we get $\mathbf{C}_S(K_3) \trianglerighteq K_3$, contradicting the structure of $\mathrm{Sp}_6(2)$ ($\mathbf{C}_{G_3 \cap G_2}(K_3) = O_2(G_2 \cap G_3)$!).

We have $\mathbf{C}_S(K_i) \subseteq K_i$ for $i = 1, 2, 4$. By (1.12) we have that $Z_1 = \Omega_1(Z(S))$. Now the pair (G_2, G_1) fulfills the assumptions of (1.14). This yields a contradiction.

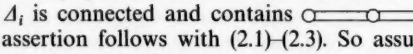
To finish the proof of (2.3) we just have to handle the case that G_1/K_1 is as in (1.8)(ii). Assume this is the case. Then $G_1/K_2 \cong G_4/K_4 \cong \mathrm{Sp}_6(2)$ and all K_i are 2-groups. Furthermore $G_3/K_3 \cong A_8$. Now the structure of G_3 shows $G_{1,3}/K_{1,3} \cong S_3 \times S_3$, but the structure of G_1 shows $G_{1,3}/K_{1,3} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \cdot 2$ by (1.8), a contradiction.

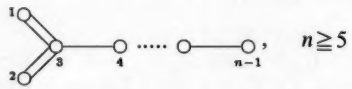
(2.4) **Proposition.** *If Δ is connected and contains* *and all other bonds are single bonds, then either $|\Delta| = 3$ and G satisfies* (1.8) *or Δ is of the shape*



$n \geqq 4$

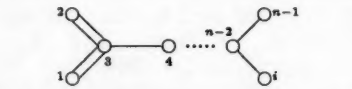
and $G_1 \cong G_2 \cong \mathrm{Sp}_{2(n-1)}(2)$, $\tilde{U}_{2(n-1)}(2)$ or $\tilde{U}_{2n-1}(2)$.

Proof. First we show that Δ is as in the assertion. We prove (2.4) by induction $|\Delta|$. We may assume that $|\Delta| > 3$. Now let i be some vertex of Δ such that the residual diagram Δ_i is connected and contains  . If $|\Delta_i| = 3$, then $|\Delta| = 4$ and the assertion follows with (2.1)–(2.3). So assume $|\Delta_i| \geqq 4$. By induction we may assume that Δ_i is of type



$n \geqq 5$

Suppose i is connected with $j \neq n-1$, then Δ_{n-1} is connected and contains , so it has to be of the same shape as Δ_i, which is only possible if $j = n-2$. Now Δ is of shape



Application of [13] to Δ_1 gives a contradiction.

So we have proved that the diagrams are as in the assertion. We now determine the groups for $|\Delta| > 4$.

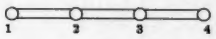
By (2.2) we have $G_{1,5,\ldots,n}/K_{1,5,\ldots,n} \cong \mathrm{Sp}_6(2)$, $\tilde{U}_6(2)$ or $\tilde{U}_7(2)$. This gives $G_1/K_1 \cong \mathrm{Sp}_{2(n-2)}(2)$, $\tilde{U}_{2(n-2)}(2)$ or $\tilde{U}_{2n-1}(2)$. The same holds for G_2/K_2. By induction we have $K_n = K_{1,n} = K_{2,n}$ and so $K_1 \subseteq K_n \supseteq K_2$. We have that $K_1 K_2/K_1$ and $K_1 K_2/K_2$ are $\{2, 3\}$-groups and so by (1.5) K_1 and K_2 are $\{2, 3\}$-groups. As $O_3(G_1 \cap G_2/K_2) = 1 = O_3(G_1 \cap G_2/K_1)$, we have that $O_3(K_1) \subseteq K_2$ and

$O_3(K_2) \subseteq K_1$ and so $O_3(K_1) = O_3(K_2) = 1$. As $O_2(K_1 K_2/K_2) \lhd G_1 \cap G_2/K_2$ and $O_2(K_1 K_2/K_2) \subseteq K_n/K_2$ we get $O_2(K_1 K_2/K_2) = 1$ ($\mathrm{Sp}_x(2)$, $U_x(2)$ do not possess such nontrivial subgroups which can be checked using the Steinberg relations). But then $O_2(K_1) \subseteq K_2$ and $O_2(K_2) \subseteq K_1$. This gives $O_2(K_1) = O_2(K_2) = 1$ and so as $O_3(K_i) = 1$ and K_i is a $\{2, 3\}$-group, we get $K_i = 1$ for $i = 1, 2$.

§ 3. Proof of the Theorem

We again assume that $\mathscr{C}$ fulfills the hypotheses of the theorem.

(3.1) **Lemma.** *Δ is not of type*



Proof. We have that G_1/K_1 and G_4/K_4 satisfy (1.8). If G_1/K_1 satisfies (1.8)(v), then $G_{1,2}/K_{1,2}$ involves $U_5(2)$ and so G_4/K_4 satisfies (1.8)(v). But from the structure of G_1 we get that $X_2/K_{1,2,4}$ involves $L_2(4)$ while from the structure of G_4 we get that it does not. Next we show

(3.1.1) *G_1/K_1 does not satisfy* (1.8)(iv).

We will prove (3.1.1) in a series of steps depending on the structure of G_4. For this we assume that G_1/K_1 satisfies (1.8)(iv).

(1) *G_4/K_4 does not satisfy* (1.8)(iv).

Suppose false. Then we get that $G_{1,2}/K_{1,2}$ and $G_{3,4}/K_{3,4}$ both involve $U_4(2)$. This shows that $U_4(2) \times L_2(4) \lesssim G_2/K_2 \lesssim \mathrm{Aut}(U_4(2) \times L_2(4))$ and $U_4(2) \times L_2(4) \lesssim G_3/K_3 \lesssim \mathrm{Aut}(U_4(2) \times L_2(4))$. Hence $K_2 K_3/K_2$ and $K_2 K_3/K_3$ are both elementary abelian of order 16. By (1.5) K_2 and K_3 are 2-groups. Now we have the following: If $G_{1,2}/K_{1,2} \cong U_4(2)$ then by (1.8) $G_{2,3}/K_{2,3} \cong A_6$ and so again by (1.8) $G_{3,4}/K_{3,4} \cong U_4(2)$. Thus in this case $G_2/K_2 \cong G_3/K_3 \cong U_4(2) \times A_5$.

If $G_{1,2}/K_{1,2} \cong U_4(2) \cdot 2$ then by (1.8) $G_{2,3}/K_{2,3} \cong S_6$ and so $G_{3,4}/K_{3,4} \cong U_4(2) \cdot 2$, hence $G_2/K_2 \cong G_3/K_3 \cong (U_4(2) \times L_2(4)) \cdot 2$. So in any case K_1 and K_4 are 2-closed. Set $Q_i = O_2(K_i)$, $i = 1, 4$. Then $|K_i/Q_i| = 3$.

Claim $\mathbf{C}_S(Q_i) \subseteq Q_i$, $i = 1, 4$. Suppose this is false. If $\mathbf{C}_S(Q_1) \supsetneq Q_1$, then $Q_1 \subseteq K_4$ as $Z(G_1 \cap G_4/K_4) = 1$. So $Q_1 \subseteq Q_4$. As $|Q_1| = |Q_4|$ we have $Q_1 = Q_4 = 1$. Now $K_3 \lhd G_1 \cap G_3$. As $O_2(G_1 \cap G_3)$ is extraspecial of order 2^7, we get $|K_3| = 2$ or 2^7. As $|S| = 2^{13}$ or 2^{14}, and $G_3/K_3 \cong U_4(2) \times L_2(4)$ or $(U_4(2) \times L_2(4)) \cdot 2$ we have a contradiction.

So we have $\mathbf{C}_S(Q_i) \subseteq Q_i$, $i = 1, 4$. As by (1.11) G_i/K_i do not possess F-modules with offending subgroup in S/K_i, we get $\Omega_1(Z(S)) \lhd G_1$ and $\Omega_1(Z(S)) \lhd G_4$. Now we apply [16; (2.2)] to the pair (X_4, G_4). This yields that Z_4 is an F_2-module for G_4 with offending subgroup in S/K_i, a contradiction to (1.11).

(2) *G_4/K_4 does not satisfy* (1.8)(iii).

Suppose G_4/K_4 satisfies (1.8)(iii). Now $K_1 \subseteq K_4$ and K_4/K_1 is a 2-group. Then by (1.5) K_4 is a 2-group. If $\mathbf{C}_S(K_1) \subseteq K_1$, then $Z(K_4) \subseteq Z(K_1)$ and so $\Omega_1(Z(K_1)) \neq \Omega_1(Z(S))$. Now by (1.11) $J(K_4) = J(K_1) \lhd \langle G_1, G_4 \rangle = G$, a contradiction.

So we may assume $\mathbf{C}(K_1) \trianglerighteq K_1$. We have $\mathbf{Z}_3 \times A_6 \lesssim G_2/K_2 \lesssim \Sigma_3 \times \Sigma_6$. Let $t \in G_2 \setminus G_1$. Then t normalizes $O^2(G_1 \cap G_2)$ and so $(K_2 \cap K_1)^t$ is centralized by $O^2(G_1 \cap G_2)$. But $Z(G_1 \cap G_2/K_1) = 1$ and so $(K_2 \cap K_1)^t \subseteq K_1 \cap K_2$. This shows $K_1 \cap K_2 \lhd \langle G_2, O^2(G_1 \cap G_4)\rangle = \langle G_2, G_1\rangle = G$. Hence $K_1 \cap K_2 = 1$, and so $|K_1| \leqq 2$. We have $[K_1, K_2] = 1$.

Claim $K_1 = 1$. Suppose this is false. Then K_1 inverts some $\rho \in X_1$, $o(\rho) = 3$. Hence $[\rho, K_2] = 1$. As $\langle \rho, K_1\rangle \lhd G_2$, we get that $\langle \rho\rangle \lhd X_1$, but $X_1/K_{4,3}$ is a parabolic in $U_4(2)$ and $U_4(2)$ possesses no element of order 3 normalized by a Sylow 2-subgroup.

So we have $K_1 = 1$. Hence K_4 is extraspecial of order 2^7. Now $G_3 \cap G_4/K_4$ acts nontrivially on K_4 and so it acts irreducibly on $K_4/Z(K_4)$. This shows $K_4 = K_3 \lhd \langle G_3, G_4\rangle = G$, a contradiction.

Now we can finish the proof of (3.1.1). We have that G_2/K_2 is involved in $S_3 \times U_4(2) \cdot 2$. Thus both K_1K_2/K_2 and K_2K_1/K_1 are 2-groups. In particular K_1 and K_2 are 2-groups by (1.5).

Let $\mathbf{C}_S(K_1) \subseteq K_1$. Then by (1.12) $Z_1 = \Omega_1(Z(S))$. Now application of [15; (2.9)] yields a module V for G_1/K_1 such that $|[V, t]| \leqq 2^5$ for $t \in Z(S/K_1)$. This contradicts (1.11).

We have $\mathbf{C}_S(K_1) \trianglerighteq K_1$. If G_4/K_4 satisfies (1.8)(i) then as $Z(G_1 \cap G_4/K_4) = 1$ we get $K_1 \subseteq K_4$. But $G_1 \cap G_4/K_1$ possesses no factor group isomorphic to $G_1 \cap G_4/K_4$ by (1.8). So we have that G_4/K_4 is as in (1.8)(ii). Now $G_2/K_2 \cong S_3 \times U_4(2)$, $S_3 \times U_4(2) \cdot 2$ or $(\mathbf{Z}_3 \times U_4(2)) \cdot 2$. In any case $G_{2,4}/K_{2,4} \cong S_3 \times S_3$. But from the structure of G_4 we get with (1.8) that $G_{2,4}/K_{2,4} \cong (\mathbf{Z}_3 \times \mathbf{Z}_3) \cdot 2$, a contradiction.

(3.1.2) *G_1/K_1 does not satisfy* (1.8)(iii).

Suppose false. Then we apply (3.1.1) to G_4/K_4. This shows that G_4/K_4 satisfies (1.8)(iii). This implies that X_2/K_4K_1 involves $L_2(4)$ while the structure of G_1 implies that X_2/K_4K_1 does not involve $L_2(4)$.

By (3.1.1) and (3.1.2) we have that G_1/K_1 and G_4/K_4 satisfy (1.8)(i) or (ii). In particular by (1.6) all K_i are 2 groups.

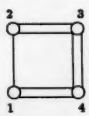
(3.1.3) *G_1/K_1 does not satisfy* (1.8)(ii).

We have that $G_3/K_3 \cong S_3 \times A_6$, $S_3 \times S_6$ or $(\mathbf{Z}_3 \times A_6) \cdot 2$. In any case $G_{1,3}/K_{1,3} \cong S_3 \times S_3$. But by (1.8) the structure of G_1 implies that $G_{1,3}/K_{1,3} \cong (\mathbf{Z}_3 \times \mathbf{Z}_3) \cdot 2$, a contradiction.

So we have shown that both G_1/K_1 and G_4/K_4 satisfy (1.8)(i). Claim $\mathbf{C}_S(K_i) \trianglerighteq K_i$ for $i = 1$ or 4. Suppose this is false. Then we may assume $Z_1 \neq \Omega_1(Z(S))$. Application of (1.12) gives a contradiction.

Without loss $\mathbf{C}_S(K_1) \trianglerighteq K_1$. As $Z(G_1 \cap G_4/K_4) = 1$ by (1.8), we get that $K_1 \subseteq K_4$. Now $K_4/K_1 \lhd G_1 \cap G_4/K_1$ and so by (1.8) either $K_1 = K_4$ or $|S: K_4| \leqq 2^5$. But we know $|S: K_4| \geqq 2^7$. This shows $K_1 = K_4 = 1$. Now by (1.8)(i) we have $G_{1,3} \subseteq \mathbf{C}(Z(S))$ and $G_{2,4} \subseteq \mathbf{C}(Z(S))$. But then $G = \langle G_{1,3}, G_{2,4}\rangle \subseteq \mathbf{C}(Z(S))$, a contradiction.

(3.2) **Lemma.** *Δ is not of shape*



Proof. Suppose false. If G_1/K_1 satisfies (1.8)(v), then also G_2/K_2 satisfies (1.8)(v). But comparing the structure of X_4/K_1 and X_4/K_2 gives a contradiction.

(3.2.1) G_1/K_1 *does not satisfy* (1.8)(iv).

Suppose this is false. We first show

(1) G_2/K_2 *does not satisfy* (1.8)(iii).

Suppose this is false. Then $G_{1,2}/K_{1,2} \cong U_4(2)$ or $U_4(2)\cdot 2$. Furthermore by (1.8) $G_{2,1,4}/K_{2,1,4} \cong S_3$ and $G_{2,3,4}/K_{2,3,4} \cong S_3$. This shows $G_4/K_4 \cong \mathrm{Sp}_6(2)$. Now $K_1 \subseteq K_2$ and K_2/K_1 is a 2-group. Thus we have that K_2 is a 2-group by (1.5). Furthermore $|K_4/K_1| = 2^{10}\cdot 3$ and so K_4 is 2-closed with $|K_4/O_2(K_4)| = 3$. Then $G_4/O_2(K_4) \cong \mathbb{Z}_3 \times \mathrm{Sp}_6(2)$, as there is no 3-part in the Schur multiplier of $\mathrm{Sp}_6(2)$. But this contradicts the fact that $G_1 \cap G_4$ involves $3\cdot A_6$. This proves (1).

(2) G_2/K_2 *does not satisfy* (1.8)(iv).

Suppose false. Then $G_{1,3,4}/K_{1,3,4} \cong L_2(4)$ or $L_2(4)\cdot 2$. This gives $G_4/K_4 \cong \tilde{U}_6(2)$ and $G_3/K_3 \cong \tilde{U}_6(2)$. Now K_3K_4/K_4 and K_3K_4/K_3 are 2-groups. Thus by (1.5) K_3 and K_4 are 2-groups. Now we get that K_2K_3/K_3 is contained in the point stabilizer of $\tilde{U}_6(2)$ and so K_2K_3/K_3 is 2-closed with 3-part of order at most 3. This shows that K_2 is 2-closed and $|K_2 : O_2(K_2)| \leqq 3$. The same holds for K_1. Set $Q_i = O_2(K_i)$, $i = 1, 2$.

Claim $\mathbf{C}_S(Q_i) \supsetneq Q_i$ for $i = 1$, 2 or 2. Suppose this is false. As by (1.11) neither G_1/K_1 nor G_2/K_2 possess an F-module, we may assume that $Z_1 = \Omega_1(Z(S))$. Then $Z_3 \neq \Omega_1(Z(S))$ and so we have that Z_3 is an F-module. By (1.1) Z_3 is the natural module. Then by (1.3) $\Omega_1(Z(S)) \lhd \langle X_1, X_4, G_1\rangle = G$, a contradiction.

Without loss $\mathbf{C}_S(Q_1) \supsetneq Q_1$. As $Z(G_1 \cap G_2/K_2) = 1$, we get $Q_1 \subseteq K_2$ and so $Q_1 = Q_2 = 1$. Hence $|S| \leqq 2^{14}$. But the structure of $\tilde{U}_6(2)$ shows $|S : K_3| \geqq 2^{15}$, a contradiction. This proves (2).

By (1) and (2) we have G_2/K_2 satisfies (1.8)(i) or (ii). In these cases $G_{2,3,4}/K_{2,3,4} \cong S_3$. On the other hand $G_{1,3,4}/K_{1,3,4} \cong L_2(4)$ or $L_2(4)\cdot 2$. So $G_4/K_4 \cong \tilde{U}_6(2)$. Then the structure of G_4 implies $G_{2,3,4}/K_{2,3,4} \cong G_{1,3,4}/K_{1,3,4}$, a contradiction.

(3.2.2) G_1/K_1 *and* G_2/K_2 *satisfy* (1.8)(i) *or* (1.8)(ii).

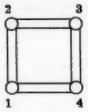
Suppose false, then by (3.2.1) we may assume that G_1/K_1 satisfies (1.8)(iii). Now $G_{1,2,4}/K_{1,2,4} \cong L_2(4)$ or $L_2(4)\cdot 2$ and so by (1.8) G_2/K_2 satisfies (1.8)(v), which contradicts (3.2.1).

Now we finish the proof. By (1.6) we have that all K_i are 2-groups. Claim $\mathbf{C}_S(K_i) \supsetneq K_i$ for $i = 1$ or 2. Suppose this is false. If G_1/K_1 and G_2/K_2 both satisfy (1.8)(i), then by (1.12) we get that $\Omega_1(Z(S)) \lhd \langle G_1, G_2\rangle = G$, a contradiction.

So without loss G_1/K_1 satisfies (1.8)(ii). Then $K_2 \subseteq K_1$ and so G_2/K_2 satisfies (1.8)(i). Otherwise $K_1 = K_2 = 1$ and so $|S| \leqq 2^4$, but as $G_3/K_3 \cong \mathrm{Sp}_6(2)$ we know $|S| \geqq 2^9$. Let $\Omega_1(Z(K_2)) \neq \Omega_1(Z(S))$, then by (1.10) $J(K_1) = J(K_2) \lhd \langle G_1, G_2\rangle = G$, a contradiction. Thus $\Omega_1(Z(K_2)) = \Omega_1(Z(S))$. Now $\Omega_1(Z(K_1)) \subseteq \Omega_1(Z(K_2)) = \Omega_1(Z(S))$ and so $\Omega_1(Z(K_1)) = \Omega_1(Z(K_2)) \lhd \langle G_1, G_2\rangle = G$, a contradiction.

So without loss $\mathbf{C}_S(K_1) \supsetneq K_1$. As $G_4/K_4 \cong \mathrm{Sp}_6(2)$ we see $|Z(G_1 \cap G_4/K_4)| = 2$ and so $|K_1K_4/K_4| \leqq 2$. In particular $|S : K_1| \geqq 2^8$ and so G_1/K_1 satisfies (1.8)(i). But as $|S : K_1K_4| = 2^8$, and $K_1K_4 \lhd G_1 \cap G_4$, we see with (1.8), that $K_4 \subseteq K_1$ and so $\mathbf{C}_S(K_4) \supsetneq K_4$. Now $K_4 \lhd \langle O^2(X_4), G_4\rangle = G$. This shows $K_4 = 1$. Then $G_3 \cong G_4 \cong \mathrm{Sp}_6(2)$ and so $|K_1| = 2$. By (1.8) applied to G_1, we get that $G_{1,3} \subseteq N(\Omega_1(Z(S)))$. But $N(\Omega_1(Z(S))) \cap G_3 = G_{1,2,3}$, a contradiction.

(3.3) **Lemma.** *If Δ is a square diagram, then Δ is of shape*



Furthermore one of the following is true

(i) $G_{i,j}/K_{i,j} \cong A_6$ *or* S_6 *for* $\{i,j\} \in \{(1,2), (2,3), (3,4), (1,4)\}$ $K_k = 1$ *for some* $k \in \{1, 2, 3, 4\}$.

(ii) $G_{i,j}/K_{i,j} \cong U_4(2)$ *or* $U_4(2) \cdot 2$ *for* $(i,j) = (1,2)$ *and* $(1,4)$. $K_2 = 1 = K_4$ *and* G_2, G_4 *satisfy* (1.8)(iv).

Proof. That Δ has the required shape follows from (3.2). Suppose first that all G_i/K_i satisfy (1.8)(i) or (ii). All we have to show is that there is some i such that $K_i = 1$. By (1.6) all K_i are 2-groups. If G_1/K_1 and G_2/K_2 both satisfy (1.8)(ii), then we get $K_1 = K_2 = 1$. So to prove (i) we just have to handle the case that G_1/K_1 satisfies (1.8)(i).

By the same argument we may assume that G_3/K_3 satisfies (1.8)(i). Let $\mathbf{C}_S(K_1) \subseteq K_1$. By (1.12) we have $\Omega_1(Z(S)) \lhd G_1$ and so $\mathbf{C}_S(K_3) \not\subseteq K_3$. Now $\Omega_1(Z(S)) \cap K_3 \lhd \langle G_1, G_4 \rangle = G$, which implies $K_3 = 1$. But then $\mathbf{C}_S(K_1) \not\subseteq K_1$, a contradiction.

So it remains to treat $\mathbf{C}_S(K_i) \not\subseteq K_i$ for $i = 1, 3$. As $K_1 \cap K_3 \lhd \langle G_1, G_3 \rangle$, we get that $K_1 \cap K_3 = 1$. Furthermore $K_1 K_3/K_1 \lhd G_1 \cap G_3$ and so $|K_1 K_3 : K_1| \leqq 2$. This shows $|K_1| \leqq 2 \geqq |K_3|$. So we may assume $|K_1| = 2 = |K_3|$ otherwise we are done. As $Z(G_1 \cap G_2/K_1) = 1$, we get $K_1 \subseteq K_2$ and so G_2 and G_4 satisfy (1.8)(ii). In particular $G_{2,4}/K_{2,4} \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) \cdot 2$. Now we have $G_{1,3} \subseteq \mathbf{C}_{G_1}(\Omega_1(Z(S)))$ and $\Omega_1(Z(S)) = K_1 K_3$. This implies that $\mathbf{C}_{K_4}(X_1) = K_3$ and $\mathbf{C}_{K_4}(X_3) = K_1$. Hence $\langle (K_1 K_3)^{G_{2,4}} \rangle = E$ is of order 16 and $[E, K_{2,4}] = 1$. But $\langle K_2, K_4 \rangle \subseteq K_{2,4}$ and $\mathbf{C}_S(K_2) = K_2$, $\mathbf{C}_S(K_4) = K_4$. In particular $E \subseteq K_2 \cap K_4$. But the size of this intersection can be determined in G_1 and so we get $|K_2 \cap K_4| = 8$, a contradiction. So we have proved (i).

Next we treat the case that one of the groups G_i/K_i does not satisfy (1.8)(i) or (ii). So without loss $G_{1,2}/K_{1,2}$ involves $U_5(2)$ or $U_4(2)$. In the former we get that both G_1/K_1 and G_2/K_2 satisfy (1.8)(v). This is impossible.

We have that $G_{1,2}/K_{1,2}$ involves $U_4(2)$. Again it is impossible that both G_1/K_1 and G_2/K_2 satisfy (1.8)(iii). So we may assume that G_2/K_2 satisfies (1.8)(iv). This implies that G_1/K_1 satisfies (1.8)(iii) and so G_4/K_4 satisfies (1.8)(iv).

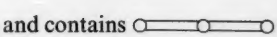
To prove (ii) we have to prove that $K_2 = K_4 = 1$. We know $K_2 \subseteq K_1 \supseteq K_4$. Furthermore K_1/K_2 and K_1/K_4 are 2-groups. Then by (1.5) K_1, K_2, K_4 are 2-groups.

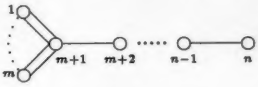
Application of (1.14) shows that $\mathbf{C}_S(K_2) \not\subseteq K_2$ and $\mathbf{C}_S(K_4) \not\subseteq K_4$ and so $K_2 \cap K_4 = 1$. By (1.8) $G_2 \cap G_4/K_2$ is an extension of a special group E of order 2^{10} by a group H with $A_5 \times S_3 \subseteq H \subseteq S_5 \times S_3$, and $|Z(E)| = 4$. As $G_4 = S\mathbf{C}_S(K_4)$, we get $[O^2(G_2 \cap G_4), K_4] = 1$. But $Z(G_2 \cap G_4/K_2) = 1$. So $K_2 = K_4 = 1$. This proves (ii).

(3.4) **Proposition.** *If Δ contains* *then the theorem holds.*

Proof. By (2.4) we may assume that the diagram Δ contains at least three double bonds. First we show

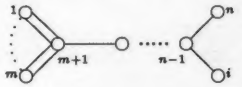
(3.4.1) *Δ is one of the diagrams of the theorem.*

Choose $i \in \Delta$ such that Δ_i is connected and contains  . By induction Δ_i belongs to the list of the theorem. Now we check the possibilities:

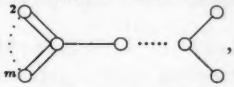
(1) Δ_i: 

$n \neq m+1$

Let i be connected with $x \in \{m+2, m, \ldots, n-1\}$. Then Δ_n is of the same type as Δ_i and so $x = n-1$ and Δ looks like

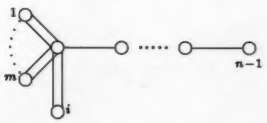


As Δ contains at least three double bounds, we get $m \geqq 3$.

Now Δ_1 is of type 

, but this diagram is not in the list, a contradiction.

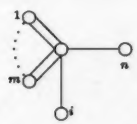
Suppose now that i is not connected with n. Then Δ_n looks like



for $n \neq m+2$

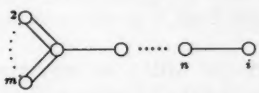
and so Δ is in the list.

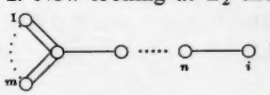
Assume $n = m+2$. If i is connected with $m+1$, then by [18] Δ is in the list or Δ looks like



But now Δ_1 is not in the list, a contradiction.

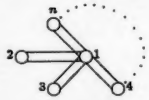
Assume now that i is connected with n. Now we look at Δ_1. As Δ_1 has to be in the list we get Δ_1 is of type

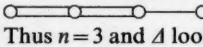
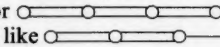
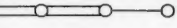


We have that i is not connected with 2. Now looking at Δ_2 shows that i is not connected with 1 and so Δ is of type 

in particular Δ is in the list.

Thus i is connected with 1. As $n \neq 2$, (3.3) implies $n \geqq 4$. Now Δ_{n-1} is connected and so Δ_{n-1} is a line. But there are no lines containing two double bonds and more than 3 nodes in the list. This handles (1).

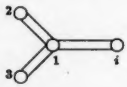
(2) *Δ_i is a star* 

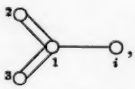
Let i be connected with $x \in \{2, \ldots, n\}$, w.l.o.g. $x = 2$. If $n = 2$, then Δ is in the list. Thus $n > 2$. By [18] i is not connected with 1. Suppose that i is not connected to any $y \in \{3, \ldots, n\}$. If $n > 3$, then $\Delta_{4,\ldots,n}$ looks like  or  both are not contained in the list. Thus $n = 3$ and Δ looks like  or , which contradicts (2.1) and (3.1).

Thus we may assume that i is connected with 3. Now Δ_2 is in the list. Let n be at least 4. Then Δ_2 is a complete bipartite graph and so Δ is a complete bipartite graph with partitions $\{1, i\} \cup \{2, \ldots, n\}$.

Let $n = 3$. Then Δ is a square and so by (3.3) Δ is in the list.

It remains the case that i is connected with 1 and no other node of Δ_i. Let $n > 3$, then Δ_2 contains at least two double bonds and so Δ is a star containing only double bonds. Let $n = 3$, then Δ is

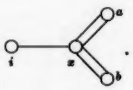


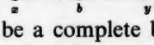
or


, both are in the list.

(3) *Δ_i is a complete bipartite graph containing only double bonds, Δ_i is not a star.*

Let $\Gamma_1 \cup \Gamma_2$ be the partition of Δ_i. Let $x \in \Gamma_1$, x connected with i. Let $y \in \Gamma_1 \setminus \{x\}$. Then Δ_y is a complete bipartite graph containing only double bounds or Δ_y looks like



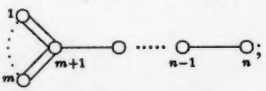
Suppose the latter. Then $|\Gamma_1| = 2 = |\Gamma_2|$. Now Δ_a is a square diagram or of type  which contradicts (2.1) and (3.3).

Let Δ_y be a complete bipartite graph containing only double bonds. Then i is connected with no element in Γ_2 and every element in $\Gamma_1 \setminus \{y\}$. If $|\Gamma_1| > 2$, then we look at Δ_x and get that Δ is a complete bipartite graph and so Δ is in the list.

Let $|\Gamma_1| = 2$. If i is connected with y we may argue as before. But now we have that Δ_y is a star and we get the assertion with (2).

It remains to prove the assertion about the groups.

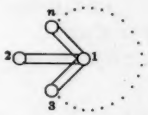
(3.4.2) *The theorem is true if Δ is of type*



; $n \neq m+1$, $m \geqq 2$:

By (2.4) we may assume $m \geqq 3$. Now by induction $K_1 = K_{3\ldots m} = K_2 = 1$. Thus $G_I \in \{\mathrm{Sp}_{2[(n-m)+1]}(2), \tilde{U}_{2[(n-m)+1]}(2), \tilde{U}_{2[(n-m)+1]+1}(2)\}$ for any $I \subseteq \{1, \ldots, m\}$, $|I| = m-1$.

(3.4.3) *The theorem is true if Δ is a star*



By (1.8) we may assume that $n \geqq 4$. We will argue by induction on n.

Let $n=4$. Assume that no G_J/K_J, $J \subseteq \{2, 3, 4\}$, $|J|=2$, involves A_6 or S_6. Then by (1.8) $K_{3,4} = K_{2,4} = K_{2,3} \trianglelefteq \langle X_1, X_2, X_3, X_4\rangle = G$, the assertion.

Assume now that any G_J/K_J, $J \subseteq \{2, 3, 4\}$, $|J|=2$ involves A_6 or S_6. Then all G_i/K_i, $i \in \{2, 3, 4\}$ satisfy (1.8)(i) or (ii). Further by (1.6) all K_i are 2-groups. If G_2/K_2 and G_3/K_3 both satisfy (1.8)(ii), then $K_2 = K_3 = 1$. So we may assume that G_2/K_3 satisfies (1.8)(i). Now by (1.14) we get that $\mathbb{C}_S(K_i) \subsetneq K_i$ for $i=2$, 3. As $Z(G_2 \cap G_3/K_2) = 1$, this implies $K_2 = K_3 = 1$.

So it remains the case that G_2/K_2 satisfies (1.8)(iv). Now by (1.8) we see that G_3/K_3 satisfies (1.8)(iv). Now K_2K_3/K_3 and K_2K_3/K_2 are 2-groups and so K_2, K_3 are 2-groups by (1.5). Again (1.14) implies that $\mathbb{C}_S(K_i) \subsetneq K_i$ for $i=2$, 3. As $|Z(G_2 \cap G_3/K_2)| = 2$ we get $|K_3K_2/K_2| \leqq 2$. Furthermore $K_2 \cap K_3 \trianglelefteq \langle G_2, G_3\rangle = G$, and so $K_2 \cap K_3 = 1$. This shows $|K_2| \leqq 2 \geqq |K_3|$.

We have to show $|K_2| = 1 = |K_3|$. Suppose this is false. As $G_{2,4}/K_{2,4}$ possesses no factorgroup $2^4 \cdot A_6$ or $2^4 \cdot S_6$, we get that G_4/K_4 satisfies (1.8)(ii). So $K_4 = K_{2,4} = K_{3,4}$. Then $Z(K_4)$ is of order 32. Now $\mathbb{C}_{Z(K_4)}(G_{2,4}) = K_2$ while $\mathbb{C}_{Z(K_4)}(G_{3,4}) = K_3$. This shows $K_2 \sim K_3$ in G_4 and so K_2 possesses 31 conjugates under G_4. Then $G_4/\mathbb{C}_{G_4}(Z(K_4)) \cong L_5(2)$. Set $Q_4 = O_2(K_4)$. Then $G_4/\mathbb{C}_{G_4}(Q_4/Z(K_4))$ is a subgroup of $L_6(2)$ involving $L_5(2)$ and $3 \cdot A_6$. This implies $G_4/\mathbb{C}_{G_4}(Q_4/Z(K_4)) \cong L_6(2)$. Set $\bar{G}_4 = G_4/\mathbb{C}_{G_4}(Q_4)$. Then $\bar{G}_4$ is an extension of a 2-group by $L_6(2)$, contradicting the fact that $\bar{G}_4/\mathbb{C}_{\bar{G}_4}(Z(K_4)) \cong L_5(2)$.

So we have shown that the assertion holds for $n=4$. By induction we have $K_2 = K_J$ for some $J \subseteq I$, $|J| = |I| - 3$, $2 \in J$. Without loss $n \in J$. Then $K_n \subseteq K_J = K_2$. If we choose K_2 of minimal order in $\{K_i | i \in \{2, \ldots, n\}\}$, then we get $K_2 = K_n = 1$, which is the assertion.

(3.4.4) *The theorem holds if Δ is a complete bipartite graph containing only double bonds, which is not a star.*

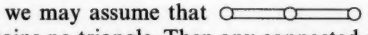
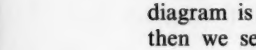
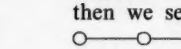
Let $\Gamma_1 \cup \Gamma_2$ be the partition of the nodes of Δ. By assumption we have $|\Gamma_1| \geqq 2 \leqq |\Gamma_2|$. By (3.3) we also may assume $|\Gamma_1| \geqq 3$. For any $i \in \Gamma$ we have that Δ_i is a complete bipartite graph. If $i \in \Gamma_j$, $j=1$ or 2, with $|\Gamma_j| > 2$, then Δ_i is not a star. There are two cases.

Case 1. For any $i \in \Gamma_1$, we have that $\Delta_{I-\{i\}}(c)$ satisfies Theorem (2)(a). Let $k \in \Gamma_1$, $j \in \Gamma_2$, then choose $i \in I_1 - \{k\}$ and so $k, j \in \Delta_i$, i.e. $G_{I-\{i,j\}}/K_{I-\{i,j\}} \cong A_6$ or S_6.

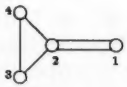
Case 2. There is some $i \in \Gamma_1$ such that $\Delta_{I-\{i\}}(c)$ is a chambersystem which satisfies Theorem (2)(c). Then Δ_i possesses a distinguished point $1 \in \Gamma_j$, $j=1$ or 2, such that $G_{I-\{1,k\}}/K_{I-\{1,k\}} \cong \Omega_6^-(2)$ or $O_6^-(2)$ for any $k \in \Gamma_{3-j} - \{i\}$ and $G_{I-\{s,t\}}/K_{I-\{s,t\}} \cong A_6$ or S_6 for any $k \in \Gamma_j \setminus \{i, 1\}$, $t \in \Gamma_{3-j} \setminus \{i\}$. As $|\Gamma_1| \geqq 3$ there is

some $x \in \Gamma_1 \setminus \{1, j\}$ and so $\Delta_{I-\{x\}}(c)$ satisfies Theorem 2(c). This shows that $G_{I-\{1,k\}}/K_{I-\{1,k\}} \cong \Omega_6^-(2)$ or $O_6^-(2)$ for any $k \in \Gamma_{3-j}$. Furthermore $G_{I-\{s,t\}}/K_{I-\{s,t\}} \cong A_6$ or S_6 for any $s \in \Gamma_1$, $t \in \Gamma_2$.

To prove the theorem it remains to show that there is some $J \subseteq I$, $|I-J|=3$, ($1 \notin J$ in Case 2) such that $K_J = 1$. For this we choose $r \in I$ ($r \in I - \{1\}$ in Case 2) such that K_r is minimal. Then $\Delta_{I-\{r\}}(c)$ satisfies Theorem (2)(a) or (c). In particular there is some $J \subseteq I$, $r \in J$ ($1 \notin J$ in Case 2), such that $K_J = K_r$. As $|I| \geqq 5$ there is some $t \in J \setminus \{r\}$. Then $K_r = K_J \geqq K_t$. By minimality we get $K_J = K_r = K_t = 1$, the assertion.

(3.5) *Proof of the theorem.* By (3.4) we may assume that  is not contained in Δ. Suppose that Δ contains no triangle. Then any connected rank 3 diagram is spherical by [18]. Now we may apply [13]. If Δ is not spherical then we see that we always have some $J \subseteq T$ such that $\Delta_{I-J}(c)$ belongs to  or  and $G_J/K_J \cong A_7$, a contradiction. So we have that Δ is spherical and G is an extension of a group of Lie type by diagonal or field automorphisms.

Suppose now that is a subdiagram of Δ. Choose Δ_1 such that Δ_1 contains and is connected. By [17] Δ_1 is a complete graph. Now for some $i \in \Delta_1$, we have that Δ_i is connected and contains a double bond. This implies that Δ is of shape



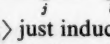
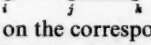
But by [18] applied to G_1, we get that $G_{1,4}/K_{1,4}$ is a solvable group, while the structure of G_4/K_4, implies that $G_{1,4}/K_{1,4}$ involves $L_3(q)$, a nonsolvable group. This contradiction proves the theorem.

§ 4. Geometries

We are going to prove the corollary. First of all we reduce the problem to a problem about chamber system.

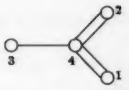
(4.1) **Lemma.** *Suppose Γ is a classical, locally finite Tits geometry over I with diagram $\Delta = \Delta(I)$ and flag-transitive automorphism group G. Then we form the chambersystem $\mathscr{C}(\Gamma)$ in the following way: The chambers are the flags of Γ of type I, two such chambers being i-adjacent if and only if they coincide in a flag of type $I-\{i\}$. Then $\mathscr{C}(\Gamma)$ is a connected classical, locally finite Tits chamber system over I with diagram Δ and G is chambertransitive on $\mathscr{C}(\Gamma)$.*

Proof [20].

Let  or  be a subdiagram of Δ. If never $\langle X_i, X_j, X_k \rangle$ just induces A_7 on the corresponding chambersystem the corollary follows from (4.1) and the theorem. In the remaining cases we need [3] and [14] to prove the corollary.

But there is a little difference in the conclusion of the corollary and the theorem in [14]. Hence we show following an idea of Th. Meixner.

(4.2) **Lemma.** *There is no classical, finite Tits geometry with diagram* Δ:



and flag transitive automorphism group G *such that* $G_i/K_i \cong A_7$ *for* $i=1, 2$.

Proof. By [14; (2.3)(a)] we know $G_1 \cong A_7 \cong G_2$ and G_3 is an extension of a group of odd order by A_6. We have $G_1 = \langle X_2, X_3, X_4 \rangle$. But as can be checked in A_7, we may generate G_1 in the following way:

$$G_1 = \langle \tilde{X}_2, X_3, X_4 \rangle,$$

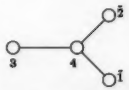
where $\langle \tilde{X}_2, X_4 \rangle \cong L_3(2)$ and $\langle \tilde{X}_2, X_3 \rangle = \tilde{X}_2 X_3$. Similarly we get

$$G_2 = \langle \tilde{X}_1, X_3, X_4 \rangle$$

with $\langle \tilde{X}_1, X_4 \rangle \cong L_3(2)$ and $\langle \tilde{X}_1, X_3 \rangle = \tilde{X}_1 X_3$. Now we determine the structure of $\langle \tilde{X}_1, \tilde{X}_2 \rangle$.

We may choose $\tilde{X}_1$, $\tilde{X}_2$ inside of G_4. But $G_4 = X_1 X_2 X_3$ and so $|G_4 : \tilde{X}_1 X_3| \leqq 3$. This shows $G_4 = \tilde{X}_1 \tilde{X}_2 X_3$ or $\tilde{X}_1 X_3 = \tilde{X}_2 X_3$. In the latter $G_1 = G_2$, a contradiction. Thus we have $G_4 = \tilde{X}_1 \tilde{X}_2 X_3$ and so as Sylow 3-subgroups of G_4 are abelian, $\langle \tilde{X}_1, \tilde{X}_2 \rangle = \tilde{X}_1 \tilde{X}_2$.

Hence $\{\tilde{X}_1, \tilde{X}_2, X_3, X_4\}$ forms a classical finite Tits chamber system with diagram



of type D_4. By [17; Theorem (3.1)] the group $\langle \tilde{X}_1, \tilde{X}_2, X_3, X_4 \rangle = \langle X_1, X_2, X_3, X_4 \rangle = G$ is an extension of $\Omega_8^+(2)$ by diagonal automorphisms. But by [14; (2.3)(a)] there is a normal subgroup K of odd order in G such that $G/K \cong A_7$, a contradiction.

(4.3) **Corollary.** *The cases* (e) *and* (f) *in the theorem of* [14] *do not occur.*

Our corollary follows immediately from the theorem, [3] and [14]. That $G \cong U_3(5)$ in the first two cases of the Corollary (4) follows from [12].

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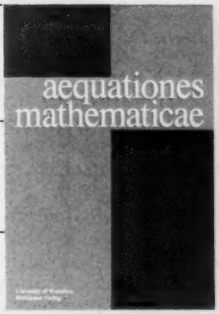

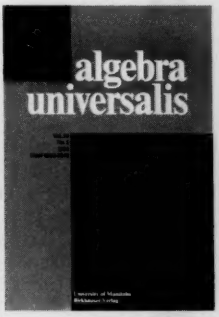

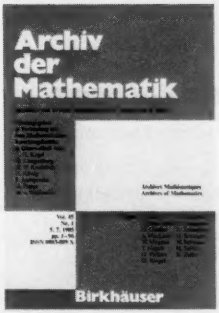

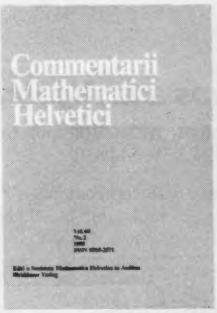

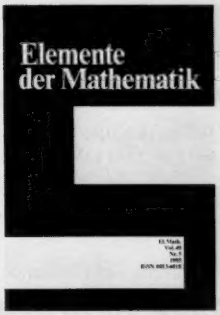

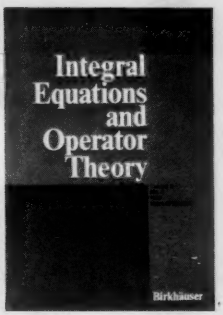

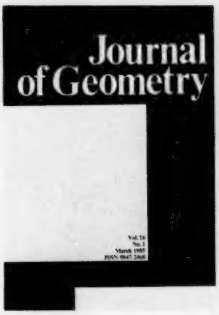

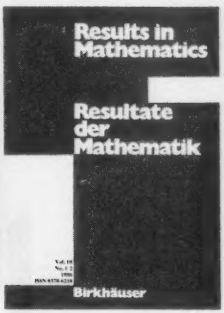

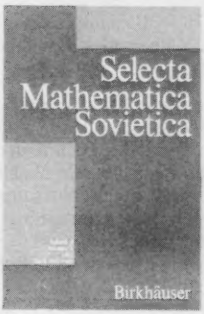

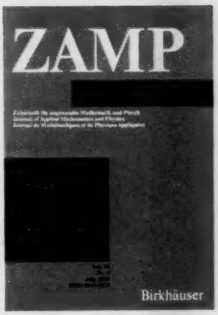

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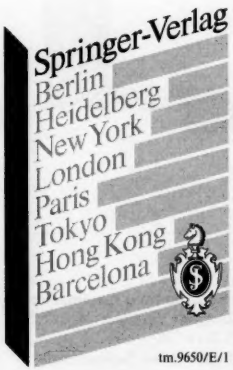

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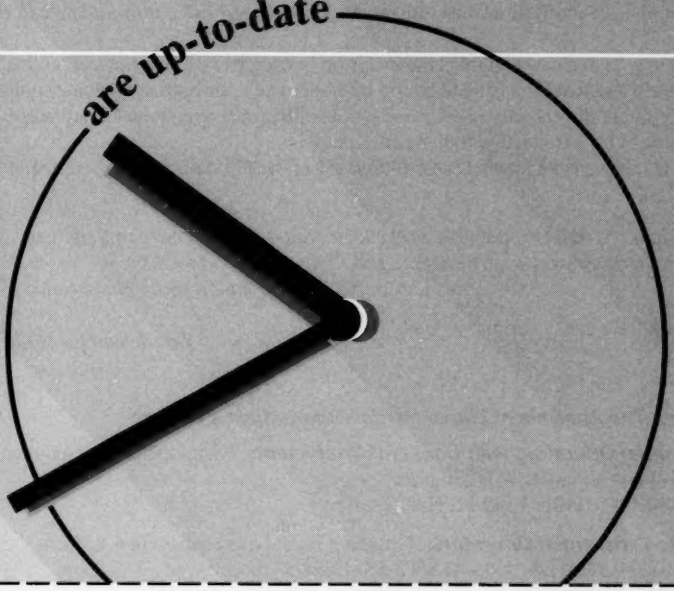

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